# EEE2035F: Signals and Systems I

## Class Test 2

24 April 2009

# SOLUTIONS

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**Name:**

**Student number:**

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### Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- Answer *all* the questions.
- You have 45 minutes.

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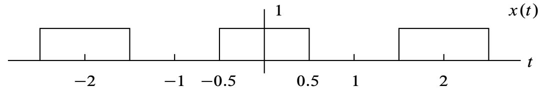
1. (5 marks) The signal




has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t},$$

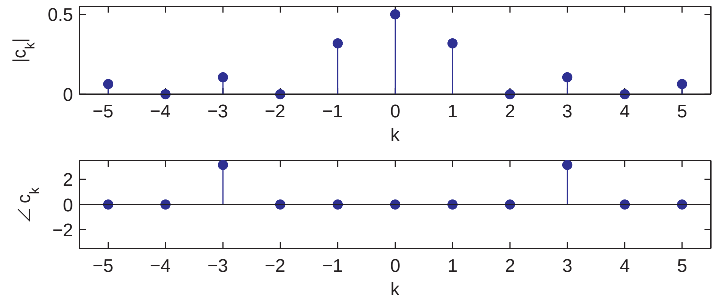
where

$$c_k = \begin{cases} 1/2 & k = 0 \\ \frac{1}{k\pi}\sin(k\pi/2) & \text{otherwise.} \end{cases}$$

(a) Plot the magnitude and phase of the Fourier series coefficients corresponding to $k = -5$ to 5.

(b) Use the information given to find a Fourier series representation for the signal $y(t) = x(t)e^{j5\pi t}$.

(a) The magnitude and phase plots are




The index $k$ indicates the harmonic multiple of the fundamental frequency $\omega_0 = \pi$.

(b) Since

$$y(t) = x(t)e^{j5\pi t} = e^{j5\pi t}\left(\sum_{k=-\infty}^{\infty} c_k e^{jk\pi t}\right) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t} e^{j5\pi t}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j(k+5)\pi t} = \sum_{m=-\infty}^{\infty} c_{m-5} e^{jm\pi t} = \sum_{k=-\infty}^{\infty} c_{k-5} e^{jk\pi t}$$

This is in the form of a Fourier series

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\pi t}$$

with $d_k = c_{k-5}$ and $\omega_0 = \pi$.

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2. (5 marks) A system has an impulse response

$$h(t) = \delta(t) - 3e^{-10t}u(t).$$

(a) Find the frequency response $H(\omega)$ of the system.

(b) What is the output of the system when the input is $x(t) = e^{j5t}$?

(a) $H(\omega) = 1 - \frac{3}{j\omega+10}$.

(b) From the definition of frequency response, the output from an input complex exponential at frequency $\omega = 5$ is

$$y(t) = H(5)e^{j5t} = \left(1 - \frac{3}{j5+10}\right)e^{j5t}.$$

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3. (5 marks) Suppose a signal has a frequency domain representation that consists of a single delta function:

$$Y(\omega) = \delta(\omega - \omega_0).$$

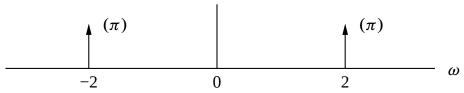
(a) Use the definition of the inverse Fourier transform to find the time domain representation $y(t)$ of the signal.

(b) Use the previous result to find and sketch the Fourier transform of the signal $x(t) = \cos(2t)$.

(a) The inverse is

$$\begin{aligned} y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega_0 t}d\omega \\ &= \frac{1}{2\pi}e^{j\omega_0 t}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)d\omega = \frac{1}{2\pi}e^{j\omega_0 t}. \end{aligned}$$

(b) Since $x(t) = \frac{1}{2}e^{j2t} + \frac{1}{2}e^{-j2t}$ and from the previous question $e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$, the Fourier transform is $X(\omega) = \pi\delta(\omega - 2) + \pi\delta(\omega + 2)$. This function is real and positive, so the magnitude is as shown below and the phase is zero.




4. (5 marks) Find the Fourier transforms of the signals below, where $u(t)$ is the unit step function:

(a) $x_1(t) = e^{-3t}u(t-5)$.

(b) $x_2(t) = t[u(t) - u(t-5)]$.

(a) Applying time shift to $e^{-3t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega+3}$ gives

$$e^{-3(t-5)}u(t-5) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega+3}e^{-j\omega 5}.$$

Since $e^{-3(t-5)}u(t-5) = e^{15}e^{-3t}u(t-5)$, the required Fourier transform is

$$X_1(\omega) = e^{-15}\frac{1}{j\omega+3}e^{-j\omega 5}.$$

(b) Applying time shift to $p_5(t) \overset{\mathcal{F}}{\longleftrightarrow} 5 sinc\frac{5\omega}{2\pi}$ gives $p_5(t-5/2) \overset{\mathcal{F}}{\longleftrightarrow} 5e^{-j\omega 5/2} sinc\frac{5\omega}{2\pi}$. Since $x_2(t) = tp_5(t-5/2)$ the multiplication by $t$ property yields

$$X_2(\omega) = j\frac{d}{d\omega} = 5j\frac{d}{d\omega}\left(e^{-j\omega 5/2} sinc\frac{5\omega}{2\pi}\right).$$

Simplifying this is simple but tedious.

# INFORMATION SHEET

## Fourier transform properties

| Property | Transform Pair/Property |
|---|---|
| Linearity | $ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$ |
| Time shift | $x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$ |
| Time scaling | $x(at) \leftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$ |
| Time reversal | $x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$ |
| Multiplication by power of $t$ | $t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$ |
| Multiplication by complex exponential | $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real |
| Multiplication by $\cos(\omega_0 t)$ | $x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$ |
| Differentiation in time domain | $\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$ |
| Integration | $\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$ |
| Convolution in time domain | $x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$ |
| Multiplication in time domain | $x(t)v(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * V(\omega)$ |
| Parseval's theorem | $\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$ |
| Parseval's theorem (special case) | $\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$ |
| Duality | $X(t) \leftrightarrow 2\pi x(-\omega)$ |

## Common Fourier Transform Pairs

| $x(t)$ | $X(\omega)$ |
|---|---|
| $1 \quad (-\infty < t < \infty)$ | $2\pi\delta(\omega)$ |
| $-0.5 + u(t)$ | $\frac{1}{j\omega}$ |
| $u(t)$ | $\pi\delta(\omega) + \frac{1}{j\omega}$ |
| $\delta(t)$ | $1$ |
| $\delta(t-c)$ | $e^{-j\omega c}$ ($c$ any real number) |
| $e^{-bt}u(t)$ | $\frac{1}{j\omega + b} \quad (b > 0)$ |
| $e^{j\omega_0 t}$ | $2\pi\delta(\omega - \omega_0)$ ($\omega_0$ any real number) |
| $p_\tau(t)$ | $\tau \operatorname{sinc} \frac{\tau\omega}{2\pi}$ |
| $\tau \operatorname{sinc} \frac{\tau t}{2\pi}$ | $2\pi p_\tau(\omega)$ |
| $\left(1 - \frac{2\lvert t\rvert}{\tau}\right) p_\tau(t)$ | $\frac{\tau}{2} \operatorname{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$ |
| $\frac{\tau}{2} \operatorname{sinc}^2 \frac{\tau t}{4\pi}$ | $2\pi\left(1 - \frac{2\lvert\omega\rvert}{\tau}\right) p_\tau(\omega)$ |
| $\cos(\omega_0 t + \theta)$ | $\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$ |
| $\sin(\omega_0 t + \theta)$ | $j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$ |

## Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1 \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$