

EEE2035F: Signals and Systems I

Class Test 2

24 April 2009

SOLUTIONS

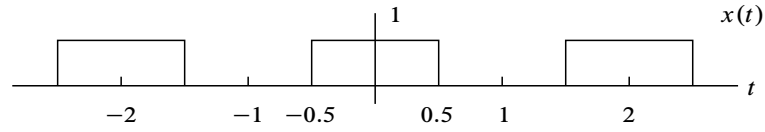
Name:

Student number:

Information

- The test is closed-book.
 - This test has *four* questions, totalling 20 marks.
 - Answer *all* the questions.
 - You have 45 minutes.
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1. (5 marks) The signal



has a Fourier series representation

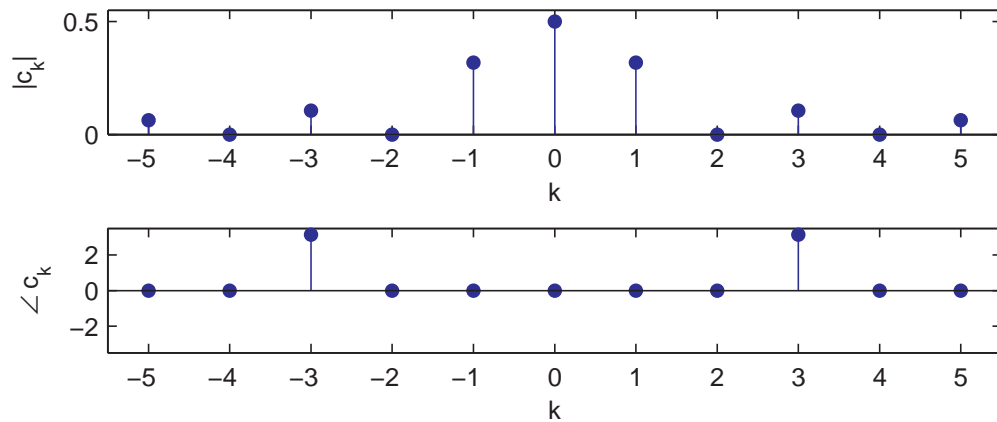
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t},$$

where

$$c_k = \begin{cases} 1/2 & k = 0 \\ \frac{1}{k\pi} \sin(k\pi/2) & \text{otherwise.} \end{cases}$$

- (a) Plot the magnitude and phase of the Fourier series coefficients corresponding to $k = -5$ to 5.
- (b) Use the information given to find a Fourier series representation for the signal $y(t) = x(t)e^{j5\pi t}$.

(a) The magnitude and phase plots are



The index k indicates the harmonic multiple of the fundamental frequency $\omega_0 = \pi$.

(b) Since

$$\begin{aligned} y(t) &= x(t)e^{j5\pi t} = e^{j5\pi t} \left(\sum_{k=-\infty}^{\infty} c_k e^{jk\pi t} \right) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t} e^{j5\pi t} \\ &= \sum_{k=-\infty}^{\infty} c_k e^{j(k+5)\pi t} = \sum_{m=-\infty}^{\infty} c_{m-5} e^{jm\pi t} = \sum_{k=-\infty}^{\infty} c_{k-5} e^{jk\pi t} \end{aligned}$$

This is in the form of a Fourier series

$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\pi t}$$

with $d_k = c_{k-5}$ and $\omega_0 = \pi$.

2. (5 marks) A system has an impulse response

$$h(t) = \delta(t) - 3e^{-10t}u(t).$$

- (a) Find the frequency response $H(\omega)$ of the system.
- (b) What is the output of the system when the input is $x(t) = e^{j5t}$?

(a) $H(\omega) = 1 - \frac{3}{j\omega + 10}$.

- (b) From the definition of frequency response, the output from an input complex exponential at frequency $\omega = 5$ is

$$y(t) = H(5)e^{j5t} = \left(1 - \frac{3}{j5 + 10}\right)e^{j5t}.$$

3. (5 marks) Suppose a signal has a frequency domain representation that consists of a single delta function:

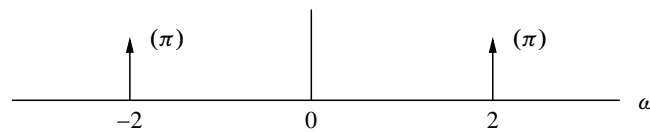
$$Y(\omega) = \delta(\omega - \omega_0).$$

- (a) Use the definition of the inverse Fourier transform to find the time domain representation $y(t)$ of the signal.
- (b) Use the previous result to find and sketch the Fourier transform of the signal $x(t) = \cos(2t)$.

- (a) The inverse is

$$\begin{aligned} y(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega_0 t} d\omega \\ &= \frac{1}{2\pi} e^{j\omega_0 t} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) d\omega = \frac{1}{2\pi} e^{j\omega_0 t}. \end{aligned}$$

- (b) Since $x(t) = \frac{1}{2}e^{j2t} + \frac{1}{2}e^{-j2t}$ and from the previous question $e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega - \omega_0)$, the Fourier transform is $X(\omega) = \pi\delta(\omega - 2) + \pi\delta(\omega + 2)$. This function is real and positive, so the magnitude is as shown below and the phase is zero.



4. (5 marks) Find the Fourier transforms of the signals below, where $u(t)$ is the unit step function:

(a) $x_1(t) = e^{-3t}u(t - 5)$.

(b) $x_2(t) = t[u(t) - u(t - 5)]$.

(a) Applying time shift to $e^{-3t}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega + 3}$ gives

$$e^{-3(t-5)}u(t-5) \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega + 3} e^{-j\omega 5}.$$

Since $e^{-3(t-5)}u(t-5) = e^{15}e^{-3t}u(t-5)$, the required Fourier transform is

$$X_1(\omega) = e^{-15} \frac{1}{j\omega + 3} e^{-j\omega 5}.$$

(b) Applying time shift to $p_5(t) \xleftrightarrow{\mathcal{F}} 5 \operatorname{sinc} \frac{5\omega}{2\pi}$ gives $p_5(t - 5/2) \xleftrightarrow{\mathcal{F}} 5e^{-j\omega 5/2} \operatorname{sinc} \frac{5\omega}{2\pi}$. Since $x_2(t) = tp_5(t - 5/2)$ the multiplication by t property yields

$$X_2(\omega) = j \frac{d}{d\omega} = 5j \frac{d}{d\omega} \left(e^{-j\omega 5/2} \operatorname{sinc} \frac{5\omega}{2\pi} \right).$$

Simplifying this is simple but tedious.

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t - c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{ a }X\left(\frac{\omega}{a}\right) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \dots$
Multiplication by complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0 \text{ real}$
Multiplication by $\cos(\omega_0 t)$	$x(t) \cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \dots$
Integration	$\int_{-\infty}^t x(\lambda) d\lambda \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega) d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) ^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t)$	$X(\omega)$
1 $(-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t - c)$	$e^{-j\omega c} \quad (c \text{ any real number})$
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \quad (\omega_0 \text{ any real number})$
$p_\tau(t)$	$\tau \text{sinc} \frac{\tau\omega}{2\pi}$
$\tau \text{sinc} \frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2 t }{\tau}\right) p_\tau(t)$	$\frac{\tau}{2} \text{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2} \text{sinc}^2 \frac{\tau t}{4\pi}$	$2\pi \left(1 - \frac{2 \omega }{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$

Trigonometric identities

$$\sin(-\theta) = -\sin(\theta) \quad \cos(-\theta) = \cos(\theta) \quad \tan(-\theta) = -\tan(\theta)$$

$$\sin^2(\theta) + \cos^2(\theta) = 1 \quad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$$

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \quad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$$

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$