EEE2035F: Signals and Systems I

Class Test 2

5 May 2008

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- There is an information sheet attached at the end of this paper.
- Answer *all* the questions.
- You have 45 minutes.

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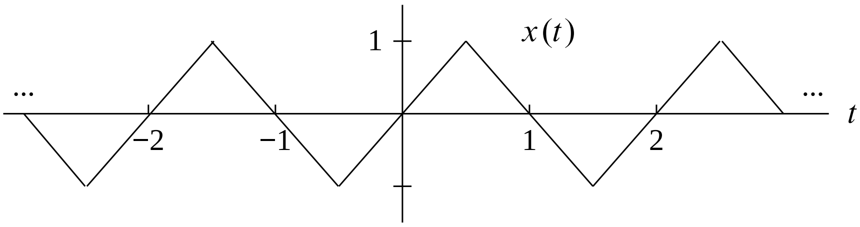
1. (5 marks) The periodic signal



has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t}$$

with

$$c_k = \begin{cases} 0 & k = 0 \\ \frac{2}{jk^2\pi^2} \sin\left(\frac{k\pi}{2}\right) & k = \pm 1, \pm 2, \ldots \end{cases}$$

(a) Plot the Fourier series coefficients (magnitude and phase) over the range $k = -4$ to $k = 4$, and

(b) Plot the Fourier series coefficients of the signal $x(t-1)$ over the same range.

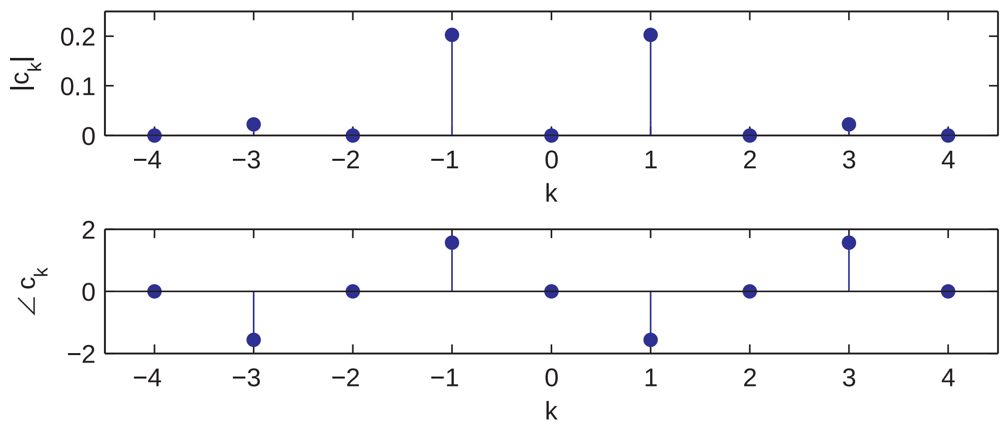
(a) For even values of k we have $c_k = 0$ since $\sin(\frac{k\pi}{2}) = 0$. Thus $c_0 = c_2 = c_4 = 0$. Also,

$$c_1 = \frac{2}{j\pi^2} = \frac{2}{e^{j\pi/2}\pi^2} = \frac{2}{\pi^2} e^{-j\pi/2}$$

and

$$c_3 = \frac{2}{j3^2\pi^2}(-1) = \frac{2}{e^{j\pi/2}9\pi^2} e^{j\pi} = \frac{2}{9\pi^2} e^{j\pi - j\pi/2} = \frac{2}{9\pi^2} e^{j\pi/2}$$

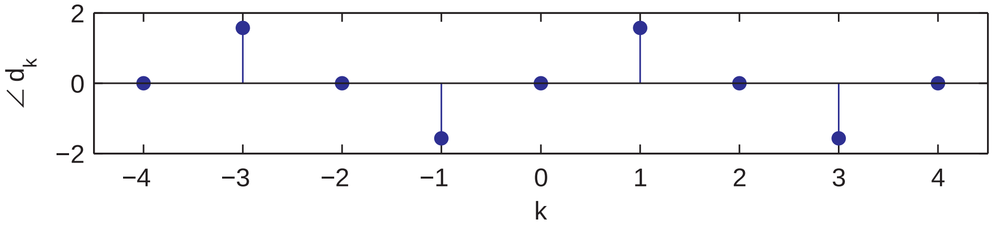
Since $x(t)$ is real, the magnitude plot is even and the phase plot odd:



The shifted signal is

$$y(t) = x(t-1) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi(t-1)} = \sum_{k=-\infty}^{\infty} c_k e^{-jk\pi} e^{jk\pi t} = \sum_{k=-\infty}^{\infty} d_k e^{jk\pi t},$$

which has Fourier coefficients $d_k = c_k e^{-jk\pi}$. Now $|d_k| = |c_k|$ and $\angle d_k = \angle c_k - k\pi$, so the magnitude is as before and the phase is



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2. (5 marks) Use the Fourier integral directly to find the transform of each of the following signals:

   (a) $x_1(t) = \delta(t) + \delta(t-2)$,

   (b) $x_2(t) = e^{2t}u(-t)$.

(a) From the definition of the Fourier transform,

$$\begin{aligned}
X(\omega) &= \int_{\infty}^{\infty} [\delta(t) + \delta(t-2)]e^{-j\omega t}\,dt = \int_{\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt + \int_{\infty}^{\infty} \delta(t-2)e^{-j\omega t}\,dt \\
&= \int_{\infty}^{\infty} \delta(t)e^{-j\omega(0)}\,dt + \int_{\infty}^{\infty} \delta(t-2)e^{-j\omega(2)}\,dt \\
&= e^{-j\omega(0)} \int_{\infty}^{\infty} \delta(t)\,dt + e^{-j\omega(2)} \int_{\infty}^{\infty} \delta(t-2)\,dt = 1 + e^{-j2\omega}.
\end{aligned}$$

(b) From the definition of the Fourier transform,

$$\begin{aligned}
X(\omega) &= \int_{\infty}^{\infty} e^{2t}u(-t)e^{-j\omega t}\,dt = \int_{\infty}^{0} e^{2t}e^{-j\omega t}\,dt = \int_{\infty}^{0} e^{(2-j\omega)t}\,dt \\
&= \left[\frac{1}{2-j\omega}e^{(2-j\omega)t}\right]_{t=-\infty}^{0} = \frac{1}{2-j\omega}\left[e^{2t}e^{-j\omega t}\right]_{t=-\infty}^{0} = \frac{1}{2-j\omega}(1-0) \\
&= \frac{1}{2-j\omega}.
\end{aligned}$$

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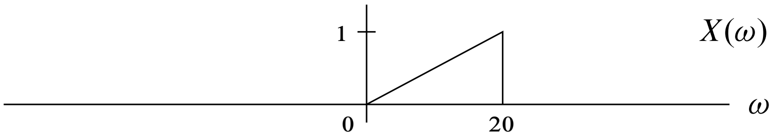
3. (5 marks) A signal $x(t)$ has the following Fourier transform (the phase of the transform is zero):



Sketch the magnitude of the Fourier transform of $x(t-5)e^{j3t}$.

Applying the time shift property with $c = 5$ to the Fourier pair $x(t) \Longleftrightarrow X(\omega)$ gives the new Fourier pair

$$x(t-5) \Longleftrightarrow X(\omega)e^{-j5\omega}.$$

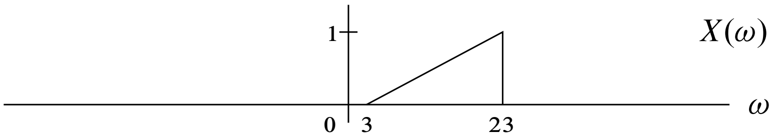
The frequency shift property with $\omega_0 = 3$ states that for any valid Fourier pair $y(t) \Longleftrightarrow Y(\omega)$ it must be true that

$$y(t)e^{j3t} \Longleftrightarrow Y(\omega-3).$$

We can apply this to the pair above by letting $y(t) = x(t-5)$ and $Y(\omega) = X(\omega)e^{-j5\omega}$:

$$x(t-5)e^{j3t} \Longleftrightarrow X(\omega-3)e^{-j5(\omega-3)}$$

The right hand side of this expression is the required Fourier transform, and its magnitude is just $X(\omega-3)$:



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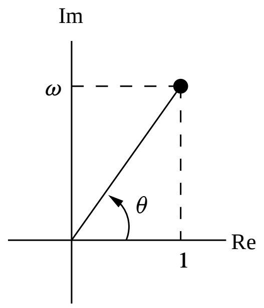
4. (5 marks) Sketch the magnitude and phase of the frequency response function

$$H(\omega) = \frac{10}{(1 + j\omega)^2}.$$

By reasoning in the complex plane



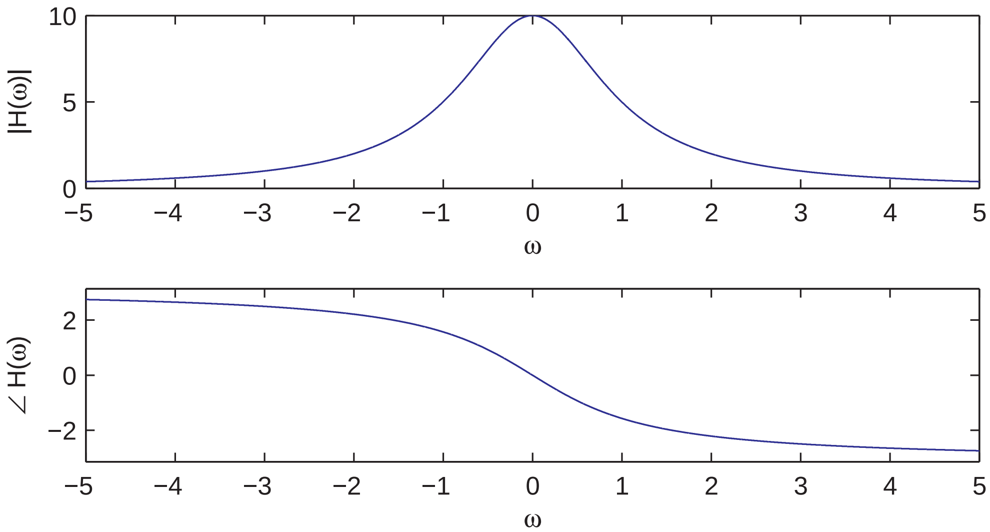
we see that the quantity $1 + j\omega$ can be written in polar form as

$$1 + j\omega = \sqrt{1 + \omega^2} e^{j \arctan(\omega)}$$

Thus we can write $H(\omega)$ in magnitude and phase form:

$$H(\omega) = \frac{10}{[\sqrt{1 + \omega^2} e^{j \arctan(\omega)}]^2} = \frac{10}{(1 + \omega^2) e^{j2 \arctan(\omega)}} = \frac{10}{1 + \omega^2} e^{-j2 \arctan(\omega)}$$

Thus $|H(\omega)| = 10/(1 + \omega^2)$ and $\angle H(\omega) = -2 \arctan(\omega)$:



INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Multiplication by complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega+\omega_0) + X(\omega-\omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|X(\omega)\|^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t)$	$X(\omega)$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c}$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega+b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$ (ω_0 any real number)
$p_\tau(t)$	$\tau \,\mathrm{sinc}\, \frac{\tau\omega}{2\pi}$
$\tau \,\mathrm{sinc}\, \frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\|t\|}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2}\mathrm{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2}\mathrm{sinc}^2 \frac{\tau t}{4\pi}$	$2\pi\left(1 - \frac{2\|\omega\|}{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega+\omega_0) + e^{j\theta}\delta(\omega-\omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega+\omega_0) - e^{j\theta}\delta(\omega-\omega_0)]$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1 \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$