EEE2035F: Signals and Systems I

Class Test 2

17 May 2007

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- There is an information sheet attached at the end of this paper.
- Answer *all* the questions.
- You have 45 minutes.

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1. (5 marks) Find the Fourier transform of the function $g(t) = u(2t) + u(t-1)$.

The function $g(t) = u(2t) + u(t-1) = u(t) + u(t-1)$ has Fourier transform

$$\begin{aligned} G(\omega) &= \mathcal{F}\{u(t)\} + \mathcal{F}\{u(t-1)\} = \mathcal{F}\{u(t)\} + e^{-j\omega}\mathcal{F}\{u(t)\} \\ &= (1+e^{-j\omega})\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] = (1+e^{-j0})\pi\delta(\omega) + \frac{(1+e^{-j\omega})}{j\omega} \\ &= 2\pi\delta(\omega) + \frac{1+e^{-j\omega}}{j\omega}. \end{aligned}$$

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2. (5 marks) Find and sketch the Fourier transform of the signal

$$y(t) = e^{-t}u(t) * \sin(2\pi t).$$

(Note that "$*$" in the above expression denotes convolution.)

Letting $x_1(t) = e^{-t}u(t)$ and $x_2(t) = \sin(2\pi t)$ we can write $y(t)$ as $y(t) = x_1(t) * x_2(t)$. In the frequency domain this corresponds to $Y(\omega) = X_1(\omega)X_2(\omega)$, and we are required to find $Y(\omega)$.

The signals in the frequency domain are

$$X_1(\omega) = \frac{1}{1 + j\omega}$$

and

$$X_2(\omega) = j\pi[\delta(\omega + 2\pi) - \delta(\omega - 2\pi)],$$

so

$$Y(\omega) = \frac{j\pi}{1 + j\omega}[\delta(\omega + 2\pi) - \delta(\omega - 2\pi)].$$

The time domain signal is the inverse transform

$$\begin{aligned} y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{j\pi}{1 + j\omega}[\delta(\omega + 2\pi) - \delta(\omega - 2\pi)]e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{j\pi}{1 + j\omega}\delta(\omega + 2\pi)e^{j\omega t}d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{j\pi}{1 + j\omega}\delta(\omega - 2\pi)]e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\left[\frac{j\pi}{1 - j2\pi}e^{-j2\pi t} - \frac{j\pi}{1 + j2\pi}e^{j2\pi t}\right] \end{aligned}$$

(Ugly way of doing this, but can massage into sinusoidal form.)

3. (5 marks) Consider a system with the following transfer function:

$$H(\omega) = \frac{j5\omega}{1 + j5\omega}.$$

(a) Sketch the magnitude of this frequency response.

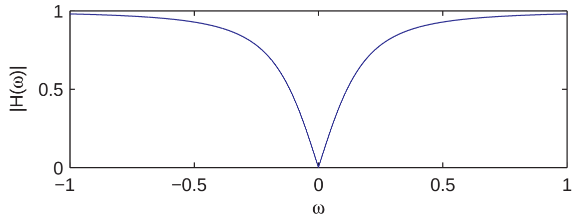
(b) What kind of filter does the system represent?

(c) What is the output of the system if the input is the signal $x(t) = \cos(2t)$?

(a) We have a ratio of complex numbers, so want numerator and denominator in polar form. For postitive ω we have $j5\omega = 5\omega e^{j\pi/2}$ and $1 + j5\omega = \sqrt{1 + (5\omega)^2}e^{j\,\mathrm{atan}(5\omega)}$, so

$$H(\omega) = \frac{j5\omega}{1 + j5\omega} = \frac{5\omega e^{j\pi/2}}{\sqrt{1 + (5\omega)^2}e^{j\,\mathrm{atan}(5\omega)}} = \frac{5\omega}{\sqrt{1 + (5\omega)^2}}e^{j(\pi/2 - \mathrm{atan}(5\omega))}$$

can similarly work out the case for negative ω. The plot of the magnitude is below:



(b) It is a highpass filter (eliminates DC and low frequencies).

(c) Since $x(t) = 1/2e^{j2t} + 1/2e^{-j2t}$ the output is

$$y(t) = 1/2H(2)e^{j2t} + 1/2H(-2)e^{-j2t} = \frac{1}{2}\frac{j5(2)}{1 + j5(2)}e^{j2t} + \frac{1}{2}\frac{j5(-2)}{1 + j5(-2)}e^{-j2t}.$$

Letting

$$\frac{j5(2)}{1 + j5(2)} = \rho e^{j\theta} = \frac{10}{\sqrt{101}}e^{j(\pi/2 - \mathrm{atan}(10))}$$

this can be written as

$$\begin{aligned} y(t) &= 1/2\rho e^{j\theta}e^{j2t} + 1/2\rho e^{-j\theta}e^{-j2t} = \rho/2(e^{j(2t+\theta)} + e^{-j(2t+\theta)}) = \rho\cos(2t + \theta) \\ &= \frac{10}{\sqrt{101}}\cos(2t + (\pi/2 - \mathrm{atan}(10))) \end{aligned}$$

4. (5 marks) Find the inverse Fourier transform of the function

$$X(\omega) = \frac{6}{3 + j\omega - j4} e^{j\omega}.$$

Start with transform pair

$$e^{-3t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{3 + j\omega}.$$

Using frequency translation

$$e^{-3t}u(t)e^{j4t} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{3 + j(\omega - 4)}.$$

Using time shift

$$e^{-3(t+1)}u(t+1)e^{j4(t+1)} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{3 + j(\omega - 4)} e^{j\omega}.$$

The required inverse transform is therefore

$$y(t) = 6e^{-3(t+1)}u(t+1)e^{j4(t+1)}.$$

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Multiplication by complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t)$	$X(\omega)$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t - c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \quad$ (ω_0 any real number)
$p_\tau(t)$	$\tau \operatorname{sinc} \frac{\tau\omega}{2\pi}$
$\tau \operatorname{sinc} \frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2} \operatorname{sinc}^2 \left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2} \operatorname{sinc}^2 \frac{\tau t}{4\pi}$	$2\pi \left(1 - \frac{2\lvert\omega\rvert}{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1 \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$