EEE2035F: Signals and Systems I

Class Test 2

19 May 2006

Name:

Student number:

Information

- The test is closed-book.
- This test has *four* questions, totalling 20 marks.
- There is an information sheet attached at the end of this paper.
- Answer *all* the questions.
- You have 45 minutes.

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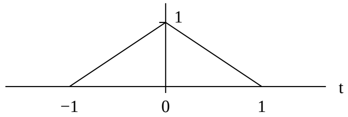
1. (5 marks) Sketch the signal

$$x(t) = \begin{cases} t+1 & -1 \le t \le 0 \\ -t+1 & 0 \le t \le 1 \end{cases}$$

and find its Fourier transform.

Sketch follows:



Letting $y(t) = dx(t)/dt$ we have $y(t) = p_1(t+1/2) - p_1(t-1/2)$, which has Fourier transform

$$Y(\omega) = \text{sinc}(\omega/2\pi)e^{j\omega/2} - \text{sinc}(\omega/2\pi)e^{-j\omega/2}.$$

From fundamental calculus the inverse of the derivative $y(t) = dx(t)/dt$ is

$$x(t) = \int_{-\infty}^{t} y(\lambda)d\lambda + c.$$

Since $\lim_{t\to-\infty} x(t) = 0$ for the given signal we must have

$$\lim_{t\to-\infty} \int_{-\infty}^{t} y(\lambda)d\lambda + c = 0,$$

so $c = 0$ (which is true for most signals we care about). Thus the left hand side of the expression for the integration property

$$\int_{-\infty}^{t} y(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega}Y(\omega) + \pi Y(0)\delta(\omega)$$

is the signal $x(t)$, and the right hand side is the required Fourier transform $X(\omega)$. Since $Y(0) = \lim_{\omega\to 0} = 1 - 1 = 0$, the solution is

$$\begin{aligned} X(\omega) &= \frac{1}{j\omega}Y(\omega) = \frac{1}{j\omega}\text{sinc}\left(\frac{\omega}{2\pi}\right)\frac{2j}{2j}(e^{j\omega/2} - e^{-j\omega}) \\ &= \frac{2j}{j\omega}\text{sinc}\left(\frac{\omega}{2\pi}\right)\sin(\omega/2) = \text{sinc}\left(\frac{\omega}{2\pi}\right)\text{sinc}\left(\frac{\omega}{2\pi}\right) = \text{sinc}^2\left(\frac{\omega}{2\pi}\right). \end{aligned}$$

Note also that we can write $x(t)$ as $(1-|t|)p_2(t)$, which is in the table of transform pairs. Alternatively, observe that $x(t) = p_1(t) * p_1(t)$ and the required transform follows immediately.

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2. (5 marks) Find $y(t) = x(t) * x(t)$ with

$$x(t) = 2\text{sinc}\left(\frac{t}{4\pi}\right).$$

(Note that reasoning in the frequency domain makes this question much simpler.)

Since convolution in time domain corresponds to multiplication in the frequency domain, the relationship between the signals is $Y(\omega) = X(\omega)X(\omega)$. From the Fourier transform tables

$$\frac{1}{2}\text{sinc}\left(\frac{t}{4\pi}\right) \Longleftrightarrow 2\pi p_{1/2}(\omega)$$

so

$$X(\omega) = 8\pi p_{1/2}(\omega)$$

and

$$Y(\omega) = X(\omega)X(\omega) = (8\pi)^2 p_{1/2}(\omega) p_{1/2}(\omega) = 32\pi[2\pi p_{1/2}(\omega)].$$

The inverse Fourier transform is obtained from the tables as

$$y(t) = 16\pi\text{sinc}\left(\frac{t}{4\pi}\right).$$

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3. (5 marks) Find the inverse Fourier transform of the signal

$$X(\omega) = e^{j5\omega}\frac{1}{4 + j\omega}.$$

Starting with the FT pair

$$x_1(t) = e^{-4t}u(t) \Longleftrightarrow X_1(\omega) = \frac{1}{4 + j\omega}$$

and applying the time shift property gives

$$x_1(t+5) = e^{-4(t+5)}u(t+5) \Longleftrightarrow X_1(\omega)e^{j\omega 5} = \frac{1}{4 + j\omega}e^{j\omega 5}.$$

Thus the required inverse transform is

$$x(t) = e^{-4(t+5)}u(t+5).$$

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4. (5 marks) Sketch each of the following transfer functions and classify them as having a lowpass, highpass, bandpass, or bandstop frequency response.

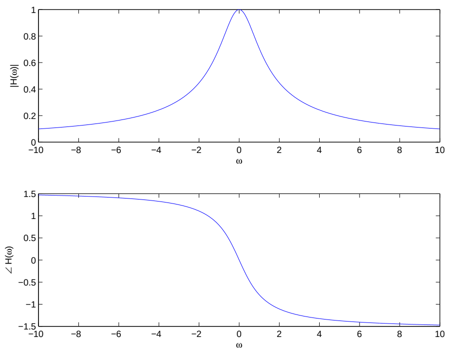
(a) $H(\omega) = \frac{1}{1+j\omega}$

(b) $H(\omega) = \frac{j\omega}{1+j\omega}$.

(a) The magnitude response for $H(\omega) = \frac{1}{1+j\omega}$ satisfies

$$|H(\omega)|^2 = H(\omega)H^*(\omega) = \frac{1}{1+j\omega}\frac{1}{1-j\omega} = \frac{1}{1+\omega^2}.$$

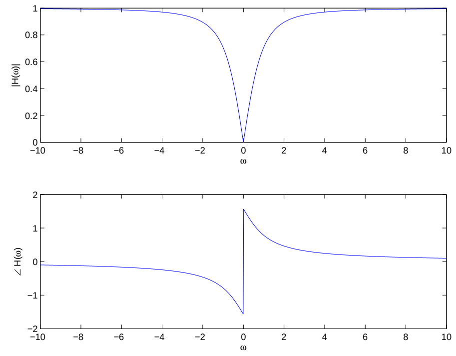
Thus $|H(\omega)|^2 = 1$ and $|H(\omega)|^2 \to 0$ as $\omega \to \infty$, so the filter has a lowpass characteristic.



(b) The magnitude response for $H(\omega) = \frac{j\omega}{1+j\omega}$ is

$$|H(\omega)|^2 = H(\omega)H^*(\omega) = \frac{j\omega}{(1+j\omega)}\frac{-j\omega}{(1-j\omega)} = \frac{\omega^2}{1+\omega^2}.$$

Now $|H(\omega)|^2 = 0$ and $|H(\omega)|^2 \to 1$ as $\omega \to \infty$, so the filter has a highpass characteristic.



INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Multiplication by complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by $\cos(\omega_0 t)$	$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$
Differentiation in time domain	$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$
Parseval's theorem (special case)	$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t)$	$X(\omega)$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \quad$ (ω_0 any real number)
$p_\tau(t)$	$\tau \operatorname{sinc} \frac{\tau\omega}{2\pi}$
$\tau \operatorname{sinc} \frac{\tau t}{2\pi}$	$2\pi p_\tau(\omega)$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2} \operatorname{sinc}^2 \left(\frac{\tau\omega}{4\pi}\right)$
$\frac{\tau}{2} \operatorname{sinc}^2 \frac{\tau t}{4\pi}$	$2\pi \left(1 - \frac{2\lvert\omega\rvert}{\tau}\right) p_\tau(\omega)$
$\cos(\omega_0 t + \theta)$	$\pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$
$\sin(\omega_0 t + \theta)$	$j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$

Trigonometric identities

$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1 \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 2\cos^2(\theta) - 1 = 1 - 2\sin^2(\theta)$

$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) \qquad \cos(\theta_1 + \theta_2) = \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)$

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$