EEE2035F Class Test

28 April 2006

Name:

Student number:

Information

- The test is closed-book.
- This test has *three* questions, totalling 30 marks.
- There is a bonus question for 5 marks.
- Answer *all* the questions.
- You have 45 minutes.

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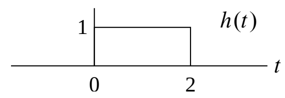
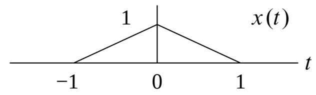
1. (10 marks) A LTI system has an impulse response



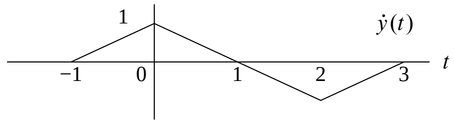
Find the response of the system to the input



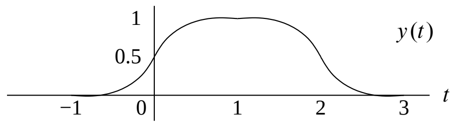
The input-output relationship for a LTI system is $y(t) = h(t) * x(t)$, where $h(t)$ is the impulse response. The required output is therefore the convolution of the two signals given. The derivative property of convolution states that $\dot{y}(t) = \dot{h}(t) * x(t)$. Since $\dot{h}(t) = \delta(t) - \delta(t-2)$ we have

$$\begin{aligned} \dot{y}(t) &= [\delta(t) - \delta(t-2)] * x(t) = \delta(t) * x(t) - \delta(t-2) * x(t) \\ &= x(t) - x(t-2), \end{aligned}$$

so



The required output is the integral of this quantity, namely



The sections of this curve are parabolic, and can be expressed as

$$y(t) = \begin{cases} \frac{1}{2}(t+1)^2 & -1 < t < 0 \\ -\frac{1}{2}(t-1)^2 + 1 & 0 < t < 2 \\ \frac{1}{2}(t-3)^2 & 2 < t < 3 \\ 0 & \text{otherwise} \end{cases}$$

if required.

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2. (10 marks) A system is described by the equation

$$y(t) = \int_{-\infty}^{t/3} x(\lambda)\,d\lambda.$$

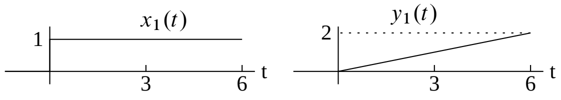
(a) Is the system causal?

(b) Is the system stable?

(c) Is the system time-invariant?

Justify your answer in each case.

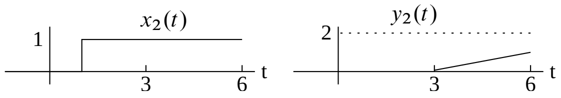
Consider the input $x_1(t) = u(t)$: the corresponding output can be found to be $y(t) = 1/3tu(t)$:



Noting that $y_1(t)$ increases without bound while the input is bounded, we can conclude that the system is not stable.

[Sincere apologies — we have not dealt with the notion of stability in this course, so this part of the question will not be marked]

Consider now $x_2(t) = u(t-1)$, which is just $x_1(t)$ delayed by 1 time unit. The corresponding output can be found to be $y_2(t) = 1/3(t-3)u(t-3)$:



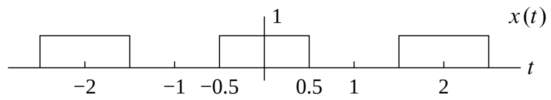
Shifting the input by one unit has resulted in the output shifting by three units, so the system is not time-invariant.

Finally, suppose we want to find the output at time $t = -1$. The system equation states that

$$y(-1) = \int_{-\infty}^{-1/3} x(\lambda)\,d\lambda,$$

so we need to know the input $x(t)$ for all times $-\infty < t < -1/3$. Some of the required values occur *later* in time than $t = -1$, so the system is not causal. (For example, to find the output $y(t)$ at time $t = -1$ we need to know the value of the input $x(t)$ at time $t = -1/3$.)

3. (10 marks) The signal



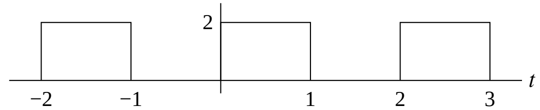
has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t},$$

where

$$c_k = \begin{cases} 1/2 & k = 0 \\ \frac{1}{k\pi}\sin(k\pi/2) & \text{otherwise.} \end{cases}$$

Use this information to find a Fourier series expansion for the signal $y(t)$ below:



The signal $y(t)$ is related to $x(t)$ via the relation $y(t) = 2x(t - {}^1\!/\!_2)$, so

$$y(t) = 2x(t - {}^1\!/\!_2) = 2\sum_{k=-\infty}^{\infty} c_k e^{jk\pi(t-1/2)} = 2\sum_{k=-\infty}^{\infty} c_k e^{jk\pi t} e^{-jk\pi/2}$$

$$= \sum_{k=-\infty}^{\infty} 2e^{-jk\pi/2} c_k e^{jk\pi t}.$$

This is in the form of a Fourier series

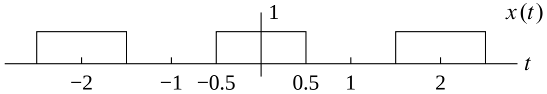
$$y(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\pi t}$$

with coefficients $d_k = 2e^{-jk\pi/2}c_k$. Therefore

$$d_k = \begin{cases} 1 & k = 0 \\ \frac{2}{k\pi}\sin(k\pi/2)e^{-jk\pi/2} & \text{otherwise} \end{cases}$$

gives the required coefficients for $y(t)$.

4. (**Bonus question:** 5 marks) The signal



has a Fourier series representation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\pi t},$$

where

$$c_k = \begin{cases} 1/2 & k = 0 \\ \frac{1}{k\pi}\sin(k\pi/2) & \text{otherwise.} \end{cases}$$

What proportion of the total signal power is contained in the frequency range $|\omega| \leq 3\pi$? Recall that Parseval's theorem states that

$$\frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt = \sum_{k=-\infty}^{\infty} |c_k|^2.$$

The average power in the signal $x(t)$ is

$$P_{\text{ave}} = \frac{1}{T}\int_{-1}^{1} x^2(t)dt = \frac{1}{2}\int_{-1/2}^{1/2} dt = \frac{1}{2}$$

watts. Since the coefficient c_k corresponds to the frequency $\omega = k\pi$, and since Parseval's theorem states that the power corresponding to c_k is $|c_k|^2$, the total power contained in the frequency range $|\omega| \leq 3\pi$ is

$$\begin{aligned} P_{(|\omega|\leq 3\pi)} &= \sum_{k=-3}^{3} |c_k|^2 \\ &= |c_{-3}|^2 + |c_{-2}|^2 + |c_{-1}|^2 + |c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 \\ &= \left|-\frac{1}{3\pi}\right|^2 + |0|^2 + \left|\frac{1}{\pi}\right|^2 + \left|\frac{1}{2}\right|^2 + |0|^2 + \left|\frac{1}{\pi}\right|^2 + \left|-\frac{1}{3\pi}\right|^2 \\ &= 0.4752. \end{aligned}$$

Thus $0.4752/0.5 \approx 95\%$ of the energy is contained within the specified frequency range.