EEE235F Class Test

20 May 2005

Name:

Student number:

Information

- The test is closed-book.
- This test has *three* questions, totalling 30 marks.
- Answer *all* the questions.
- You have 45 minutes.
- There is an information sheet attached to the end of this question paper.

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1. (10 marks) Use time-domain convolution to find the inverse Fourier transform of

$$X(\omega) = \frac{1}{(a + j\omega)^2}.$$

The signal

$$X(\omega) = \frac{1}{(a + j\omega)} \frac{1}{(a + j\omega)} = X_1(\omega) X_1(\omega)$$

where $X_1(\omega)$ has inverse transform (from tables)

$$x_1(t) = e^{-at} u(t).$$

Multiplication in the frequency domain corresponds to convolution in the time domain, so

$$x(t) = x_1(t) * x_1(t) = \int_{-\infty}^{\infty} x_1(\tau) x_1(t - \tau) d\tau = \int_{-\infty}^{\infty} e^{-a\tau} u(\tau) e^{-a(t-\tau)} u(t - \tau) d\tau$$
$$= \int_0^t e^{-a\tau} e^{-a(t-\tau)} d\tau = e^{-at} \int_0^t d\tau = t e^{-at} u(t).$$

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2. (10 marks) A time signal $x(t)$ has the Fourier transform $4/(4+\omega^2)$. Find the Fourier transforms of the following signals:

   (a) $x_1(t) = x(2t-3)$

   (b) $x_2(t) = \int_{-\infty}^{t} x(\tau) d\tau$.

(a) Given

$$x(t) \longleftrightarrow \frac{4}{4+\omega^2},$$

frequency scaling gives

$$x(2t) \longleftrightarrow \frac{1}{2}\frac{4}{4+(\frac{\omega}{2})^2} = \frac{2}{4+(\frac{\omega}{2})^2}.$$

Finally a time shift of $3/2$ gives the required transform pair:

$$x(2t-3) = x(2(t-3/2)) \longleftrightarrow \frac{2}{4+(\frac{\omega}{2})^2} e^{-j\omega 3/2}.$$

(b) The integration property states that

$$\int_{-\infty}^{t} x(\lambda) d\lambda \longleftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega),$$

so

$$\begin{aligned} x_2(t) = \int_{-\infty}^{t} x(\lambda) d\lambda &\longleftrightarrow \frac{1}{j\omega}\frac{4}{4+\omega^2} + \pi \frac{4}{4+0^2}\delta(\omega) \\ &\longleftrightarrow \frac{1}{j\omega}\frac{4}{4+\omega^2} + \pi\delta(\omega) \end{aligned}$$

3. (10 marks) Find the inverse Fourier transform of the signal

$$G(\omega) = \frac{1}{1+j\omega}\cos(2\omega)e^{-j5\omega}.$$

We can write

$$G(\omega) = \frac{1}{1+j\omega}\frac{1}{2}\left(e^{j2\omega} + e^{-j2\omega}\right)e^{-j5\omega} = \frac{1}{2}\frac{1}{1+j\omega}e^{-j3\omega} + \frac{1}{2}\frac{1}{1+j\omega}e^{-j7\omega}.$$

Now

$$e^{-t}u(t) \longleftrightarrow \frac{1}{1+j\omega},$$

so

$$\frac{1}{2}e^{-t}u(t) \longleftrightarrow \frac{1}{2}\frac{1}{1+j\omega}$$

and

$$\frac{1}{2}e^{-(t-3)}u(t-3) \longleftrightarrow \frac{1}{2}\frac{1}{1+j\omega}e^{-j3\omega}$$
$$\frac{1}{2}e^{-(t-7)}u(t-7) \longleftrightarrow \frac{1}{2}\frac{1}{1+j\omega}e^{-j7\omega}$$

Therefore the inverse is

$$g(t) = \frac{1}{2}e^{-(t-3)}u(t-3) + \frac{1}{2}e^{-(t-7)}u(t-7).$$

INFORMATION SHEET

Fourier transform properties

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Time shift	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X(\frac{\omega}{a}) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Multiplication by complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Differentiation in time domain	$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda) d\lambda \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
Convolution in time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)dt \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X(\omega)} V(\omega) d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Common Fourier Transform Pairs

$x(t)$	$X(\omega)$
$1 \quad (-\infty < t < \infty)$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$\frac{1}{j\omega}$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$\delta(t)$	1
$\delta(t-c)$	$e^{-j\omega c} \quad$ (c any real number)
$e^{-bt}u(t)$	$\frac{1}{j\omega + b} \quad (b > 0)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0) \qquad$ (ω_0 any real number)
$p_\tau(t)$	$\tau \operatorname{sinc} \frac{\tau\omega}{2\pi}$
$\left(1 - \frac{2\lvert t\rvert}{\tau}\right) p_\tau(t)$	$\frac{\tau}{2} \operatorname{sinc}^2 \left(\frac{\tau\omega}{4\pi}\right)$