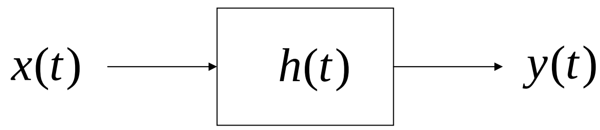
Chapter 5
Frequency Domain Analysis of Systems

CT, LTI Systems

- Consider the following CT LTI system:



- Assumption: the impulse response $h(t)$ is *absolutely integrable,* i.e.,

$$\int_{\mathbb{R}} |h(t)|\, dt < \infty$$

(this has to do with system stability (ECE 352))

Response of a CT, LTI System to a Sinusoidal Input

- What's the response $y(t)$ of this system to the input signal

$$x(t) = A\cos(\omega_0 t + \theta), \;\; t \in \mathbb{R} \quad ?$$

- We start by looking for the response $y_c(t)$ of the same system to

$$x_c(t) = Ae^{j(\omega_0 t + \theta)} \quad t \in \mathbb{R}$$

Response of a CT, LTI System to a Complex Exponential Input

- The output is obtained through convolution as

$$y_c(t) = h(t) * x_c(t) = \int_{\mathbb{R}} h(\tau) x_c(t-\tau) d\tau =$$

$$= \int_{\mathbb{R}} h(\tau) A e^{j(\omega_0 (t-\tau)+\theta)} d\tau =$$

$$= \underbrace{A e^{j(\omega_0 t+\theta)}}_{x_c(t)} \int_{\mathbb{R}} h(\tau) e^{-j\omega_0 \tau} d\tau =$$

$$= x_c(t) \int_{\mathbb{R}} h(\tau) e^{-j\omega_0 \tau} d\tau$$

The Frequency Response of a CT, LTI System

- By defining

$$H(\omega) = \int_{\mathbb{R}} h(\tau) e^{-j\omega\tau} d\tau$$

$H(\omega)$ is the frequency response of the CT, LTI system = Fourier transform of *h*(*t*)

it is

$$y_c(t) = H(\omega_0) x_c(t) =$$

$$= H(\omega_0) A e^{j(\omega_0 t + \theta)}, \quad t \in \mathbb{R}$$

- Therefore, the response of the LTI system to a complex exponential is another complex exponential with the same frequency ω_0

Analyzing the Output Signal $y_c(t)$

- Since $H(\omega_0)$ is in general a complex quantity, we can write

$$y_c(t) = H(\omega_0) A e^{j(\omega_0 t + \theta)} =$$

$$= |H(\omega_0)| e^{j \arg H(\omega_0)} A e^{j(\omega_0 t + \theta)} =$$

$$= \underbrace{A |H(\omega_0)|}_{\text{output signal's magnitude}} e^{j(\omega_0 t + \underbrace{\theta + \arg H(\omega_0)}_{\text{output signal's phase}})}$$

Response of a CT, LTI System to a Sinusoidal Input

- With Euler's formulas we can express

$$x(t) = A\cos(\omega_0 t + \theta)$$

as

$$x(t) = \Re(x_c(t)) = \tfrac{1}{2}(x_c(t) + x_c^*(t))$$

and, by exploiting linearity, it is

$$y(t) = \Re(y_c(t)) = \tfrac{1}{2}(y_c(t) + y_c^*(t)) =$$

$$= A\,|\,H(\omega_0)\,|\cos\left(\omega_0 t + \theta + \arg H(\omega_0)\right)$$

Response of a CT, LTI System to a Sinusoidal Input – Cont'd

- Thus, the response to

$$x(t) = A\cos(\omega_0 t + \theta)$$

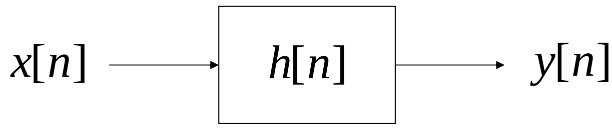
is

$$y(t) = A\,|H(\omega_0)|\cos\left(\omega_0 t + \theta + \arg H(\omega_0)\right)$$

which is also a sinusoid with the same frequency ω_0 but with the amplitude scaled by the factor $|H(\omega_0)|$ and with the phase shifted by amount $\arg H(\omega_0)$

DT, LTI Systems

- Consider the following DT, LTI system:



- The I/O relation is given by

$$y[n] = h[n] * x[n]$$

Response of a DT, LTI System to a Complex Exponential Input

- If the input signal is

$$x_c[n] = Ae^{j(\omega_0 n + \theta)} \quad n \in \mathbb{Z}$$

- Then the output signal is given by

$$y_c[n] = H(\omega_0)x_c[n] =$$

$$= H(\omega_0)Ae^{j(\omega_0 n + \theta)}, \quad n \in \mathbb{Z}$$

where

$$H(\omega) = \sum_{k \in \mathbb{Z}} h[k]e^{-j\omega k}, \quad \omega \in \mathbb{R}$$

$H(\omega)$ is the frequency response of the DT, LTI system = DT Fourier transform (DTFT) of $h[n]$

Response of a DT, LTI System to a Sinusoidal Input

- If the input signal is

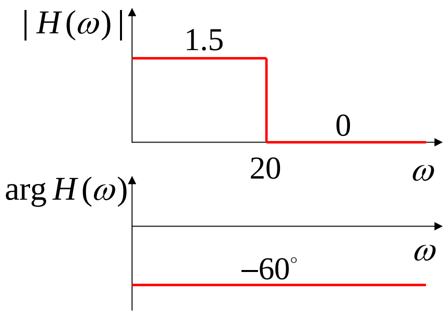
$$x[n] = A\cos(\omega_0 n + \theta) \quad n \in \mathbb{Z}$$

- Then the output signal is given by

$$y[n] = A\,|H(\omega_0)|\cos\left(\omega_0 n + \theta + \arg H(\omega_0)\right)$$

Example: Response of a CT, LTI System to Sinusoidal Inputs

- Suppose that the frequency response of a CT, LTI system is defined by the following specs:



$$|H(\omega)| = \begin{cases} 1.5, & 0 \le \omega \le 20, \\ 0, & \omega > 20, \end{cases}$$

$$\arg H(\omega) = -60^\circ, \forall \omega$$

Example: Response of a CT, LTI System to Sinusoidal Inputs – Cont'd

- If the input to the system is

$$x(t) = 2\cos(10t + 90^\circ) + 5\cos(25t + 120^\circ)$$

- Then the output is

$$y(t) = 2\,|H(10)|\cos(10t + 90^\circ + \arg H(10)) +$$
$$+ 5\,|H(25)|\cos(25t + 120^\circ + \arg H(25)) =$$
$$= 3\cos(10t + 30^\circ)$$

Example: Frequency Analysis of an RC Circuit

- Consider the RC circuit shown in figure

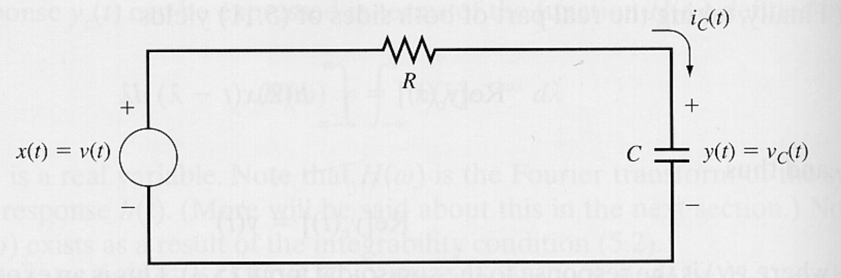


Figure 5.1 *RC* circuit in Example 5.2.

Example: Frequency Analysis of an RC Circuit – Cont'd

- From ENGR 203, we know that:
  1. The complex impedance of the capacitor is equal to $1/sC$ where $s = \sigma + j\omega$
  2. If the input voltage is $x_c(t) = e^{st}$, then the output signal is given by

$$y_c(t) = \frac{1/sC}{R + 1/sC} e^{st} = \frac{1/RC}{s + 1/RC} e^{st}$$

Example: Frequency Analysis of an RC Circuit – Cont'd

- Setting $s = j\omega_0$, it is

$$x_c(t) = e^{j\omega_0 t} \quad \text{and} \quad y_c(t) = \frac{1/RC}{j\omega_0 + 1/RC} e^{j\omega_0 t}$$

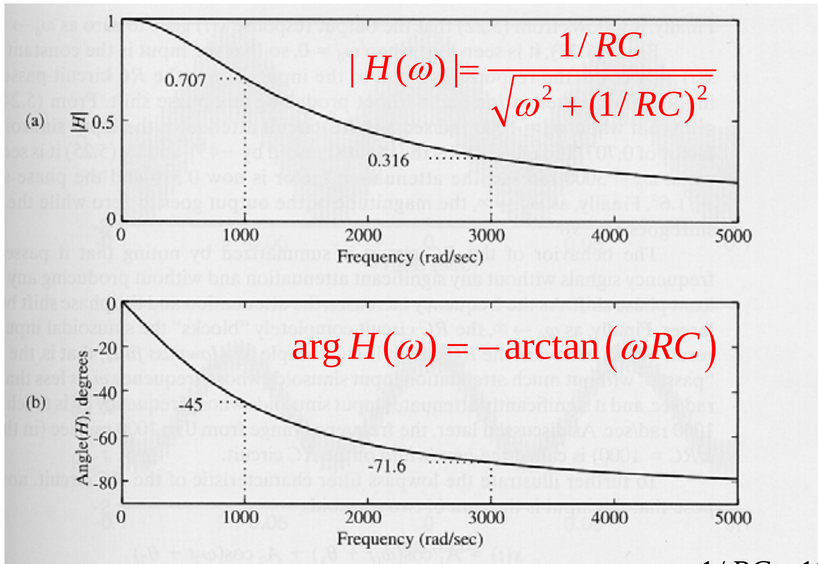
whence we can write

$$y_c(t) = H(\omega_0) x_c(t)$$

where

$$H(\omega) = \frac{1/RC}{j\omega + 1/RC}$$

Example: Frequency Analysis of an RC Circuit – Cont'd



$$|H(\omega)| = \frac{1/RC}{\sqrt{\omega^2 + (1/RC)^2}}$$

$$\arg H(\omega) = -\arctan\left(\omega RC\right)$$

$1/RC = 1000$

Example: Frequency Analysis of an RC Circuit – Cont'd

- The knowledge of the frequency response $H(\omega)$ allows us to compute the response $y(t)$ of the system to any sinusoidal input signal

$$x(t) = A\cos(\omega_0 t + \theta)$$

since

$$y(t) = A\,|H(\omega_0)|\cos\left(\omega_0 t + \theta + \arg H(\omega_0)\right)$$

Example: Frequency Analysis of an RC Circuit – Cont'd

- Suppose that $1/RC = 1000$ and that

$$x(t) = \cos(100t) + \cos(3000t)$$

- Then, the output signal is

$$\begin{aligned} y(t) &= |H(100)|\cos(100t + \arg H(100)) + \\ &+ |H(3000)|\cos(3000t + \arg H(3000)) = \\ &= 0.9950\cos(100t - 5.71^\circ) + 0.3162\cos(3000t - 71.56^\circ) \end{aligned}$$

Example: Frequency Analysis of an RC Circuit – Cont'd

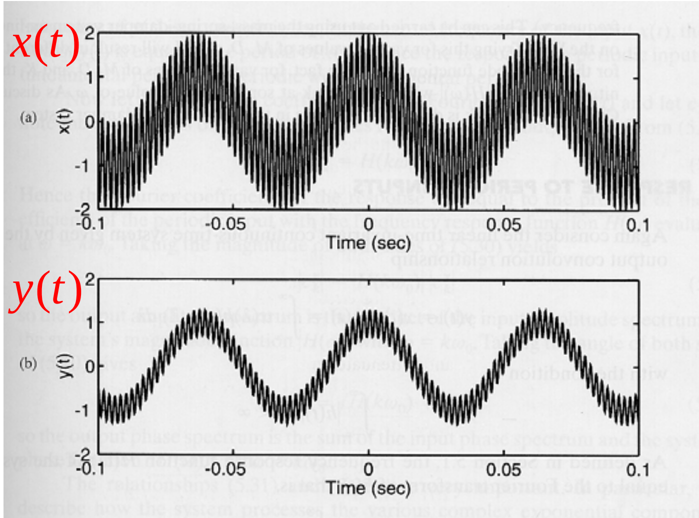


Figure 5.3 (a) Input and (b) output of *RC* circuit when $1/RC = 1000$.

Example: Frequency Analysis of an RC Circuit – Cont'd

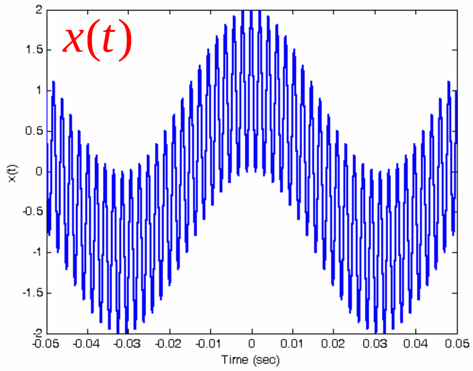
- Suppose now that

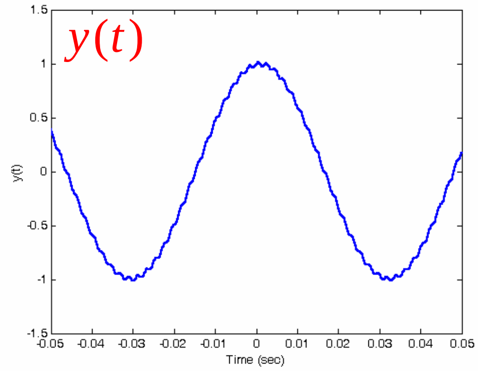
$$x(t) = \cos(100t) + \cos(50{,}000t)$$

- Then, the output signal is

$$y(t) = |H(100)|\cos(100t + \arg H(100)) +$$
$$+ |H(50{,}000)|\cos(50{,}000t + \arg H(50{,}000)) =$$
$$= 0.9950\cos(100t - 5.71^\circ) + 0.0200\cos(50{,}000t - 88.85^\circ)$$

Example: Frequency Analysis of an RC Circuit – Cont'd





The RC circuit behaves as a lowpass filter, by letting low-frequency sinusoidal signals pass with little attenuation and by significantly attenuating high-frequency sinusoidal signals

Response of a CT, LTI System to Periodic Inputs

- Suppose that the input to the CT, LTI system is a periodic signal $x(t)$ having period T
- This signal can be represented through its Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t}, \quad t \in \mathbb{R}$$

where

$$c_k^x = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt, \quad k \in \mathbb{Z}$$

Response of a CT, LTI System to Periodic Inputs – Cont'd

- By exploiting the previous results and the linearity of the system, the output of the system is

$$y(t) = \sum_{k=-\infty}^{\infty} H(k\omega_0) c_k^x e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{|H(k\omega_0)||c_k^x|}_{|c_k^y|} e^{j(k\omega_0 t + \underbrace{\arg(c_k^x) + \arg H(k\omega_0)}_{\arg c_k^y})} =$$

$$= \sum_{k=-\infty}^{\infty} |c_k^y| e^{j(k\omega_0 t + \arg(c_k^y))} = \sum_{k=-\infty}^{\infty} c_k^y e^{jk\omega_0 t}, \qquad t \in \mathbb{R}$$

Example: Response of an RC Circuit to a Rectangular Pulse Train

- Consider the RC circuit

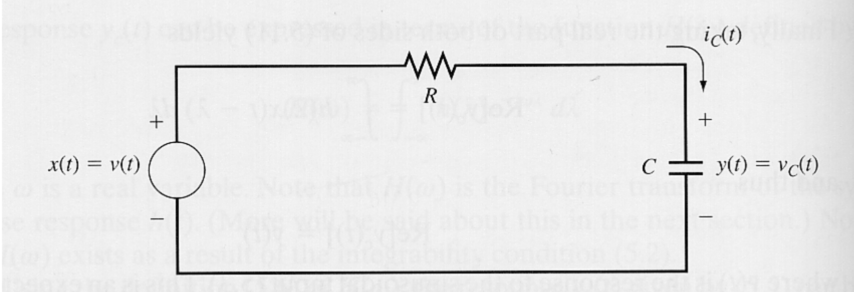


Figure 5.1 *RC* circuit in Example 5.2.

with input $x(t) = \sum_{n \in \mathbb{Z}} \operatorname{rect}(t - 2n)$

Example: Response of an RC Circuit to a Rectangular Pulse Train – Cont'd

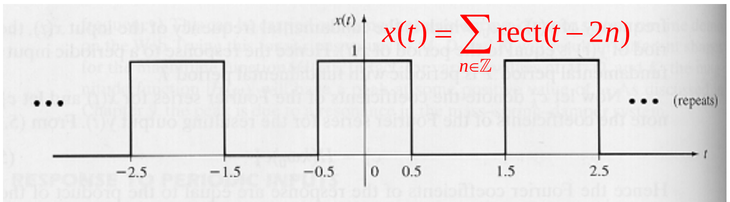


Figure 5.4 Periodic input signal in Example 5.4.

$$x(t) = \sum_{n \in \mathbb{Z}} \mathrm{rect}(t - 2n)$$

- We have found its Fourier series to be

$$x(t) = \sum_{k \in \mathbb{Z}} c_k^x e^{jk\pi t}, \quad t \in \mathbb{R}$$

with

$$c_k^x = \frac{1}{2} \mathrm{sinc}\left(\frac{k}{2}\right)$$

Example: Response of an RC Circuit to a Rectangular Pulse Train – Cont'd

- Magnitude spectrum $|c_k^x|$ of input signal $x(t)$

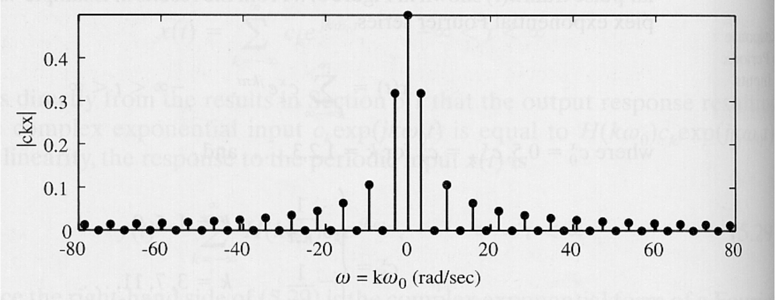


Figure 5.5 Amplitude spectrum of periodic input in Example 5.4.

Example: Response of an RC Circuit to a Rectangular Pulse Train – Cont'd

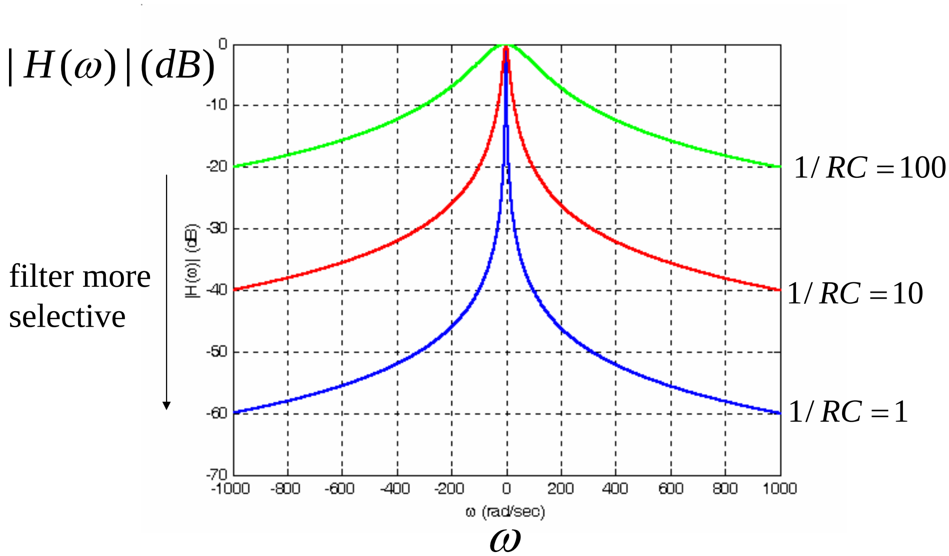
- The frequency response of the RC circuit was found to be

$$H(\omega) = \frac{1/RC}{j\omega + 1/RC}$$

- Thus, the Fourier series of the output signal is given by

$$y(t) = \sum_{k=-\infty}^{\infty} H(k\omega_0) c_k^x e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k^y e^{jk\omega_0 t}$$

Example: Response of an RC Circuit to a Rectangular Pulse Train – Cont'd



Example: Response of an RC Circuit to a Rectangular Pulse Train – Cont'd

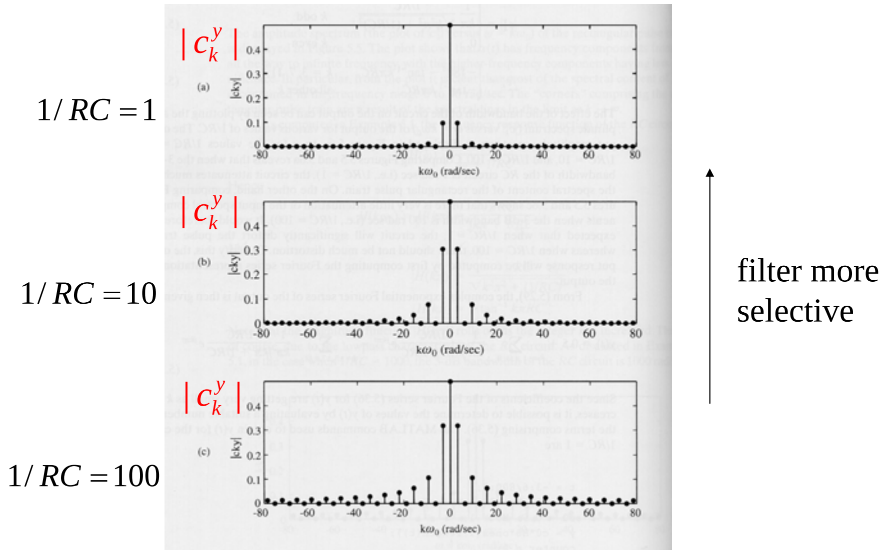


Figure 5.6 Amplitude spectrum of output when (a) $1/RC = 1$; (b) $1/RC = 10$; (c) $1/RC = 100$.

Example: Response of an RC Circuit to a Rectangular Pulse Train – Cont'd

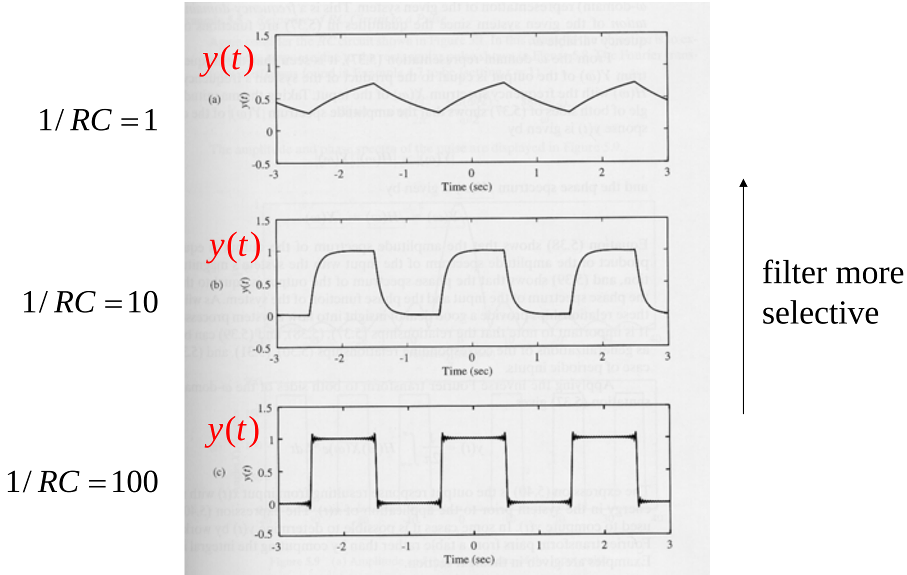
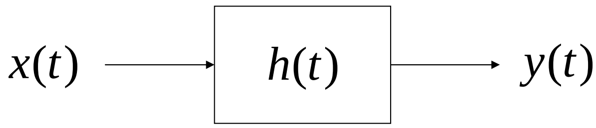


Figure 5.7 Plot of output when (a) $1/RC = 1$; (b) $1/RC = 10$; (c) $1/RC = 100$.

Response of a CT, LTI System to Aperiodic Inputs

- Consider the following CT, LTI system



- Its I/O relation is given by

$$y(t) = h(t) * x(t)$$

which, in the frequency domain, becomes

$$Y(\omega) = H(\omega)X(\omega)$$

Response of a CT, LTI System to Aperiodic Inputs – Cont'd

- From $Y(\omega) = H(\omega)X(\omega)$, the magnitude spectrum of the output signal $y(t)$ is given by

$$|Y(\omega)| = |H(\omega)||X(\omega)|$$

and its phase spectrum is given by

$$\arg Y(\omega) = \arg H(\omega) + \arg X(\omega)$$

Example: Response of an RC Circuit to a Rectangular Pulse

- Consider the RC circuit

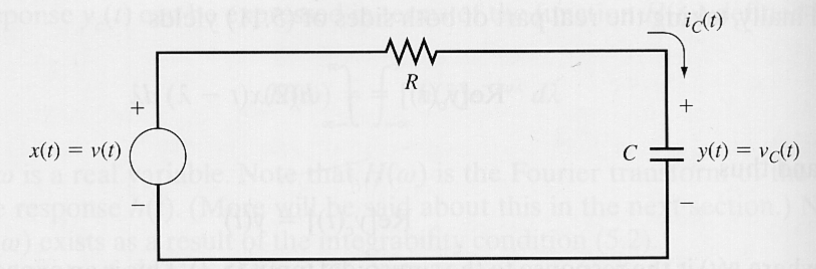
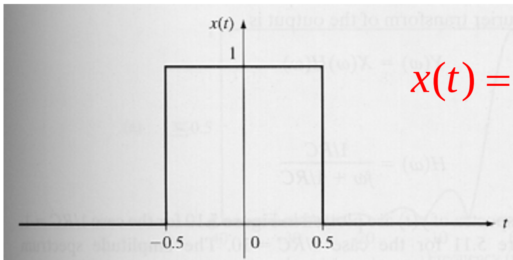


Figure 5.1 *RC* circuit in Example 5.2.

with input $x(t) = \text{rect}(t)$

Example: Response of an RC Circuit to a Rectangular Pulse – Cont'd



$$x(t) = \mathrm{rect}(t)$$

Figure 5.8 Input pulse in Example 5.5.

- The Fourier transform of $x(t)$ is

$$X(\omega) = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

Example: Response of an RC Circuit to a Rectangular Pulse – Cont'd

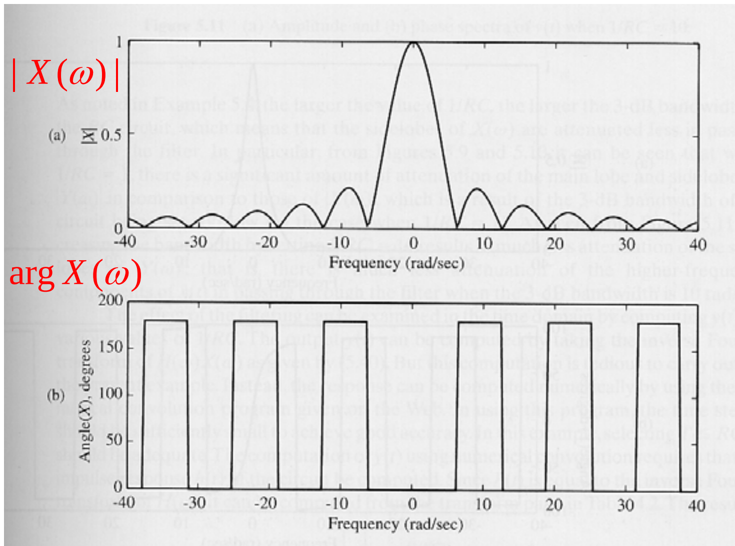


Figure 5.9 (a) Amplitude and (b) phase spectra of the input pulse.

Example: Response of an RC Circuit to a Rectangular Pulse – Cont'd

$1/RC = 1$

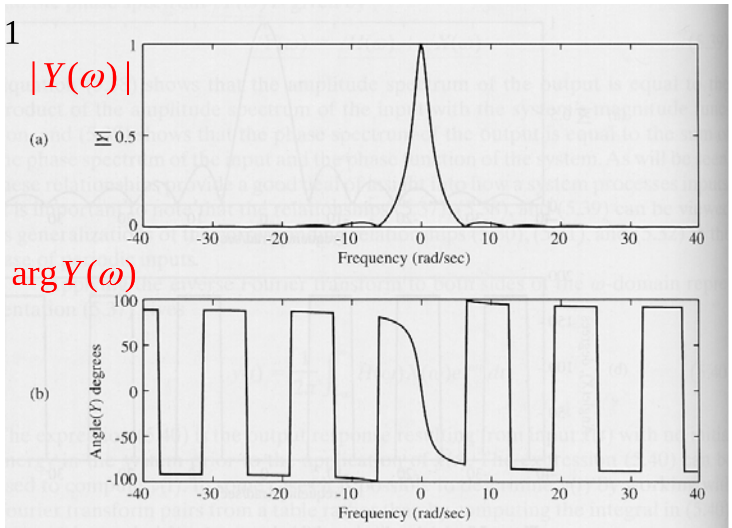


Figure 5.10 (a) Amplitude and (b) phase spectra of $y(t)$ when $1/RC = 1$.

Example: Response of an RC Circuit to a Rectangular Pulse – Cont'd

$1/RC = 10$

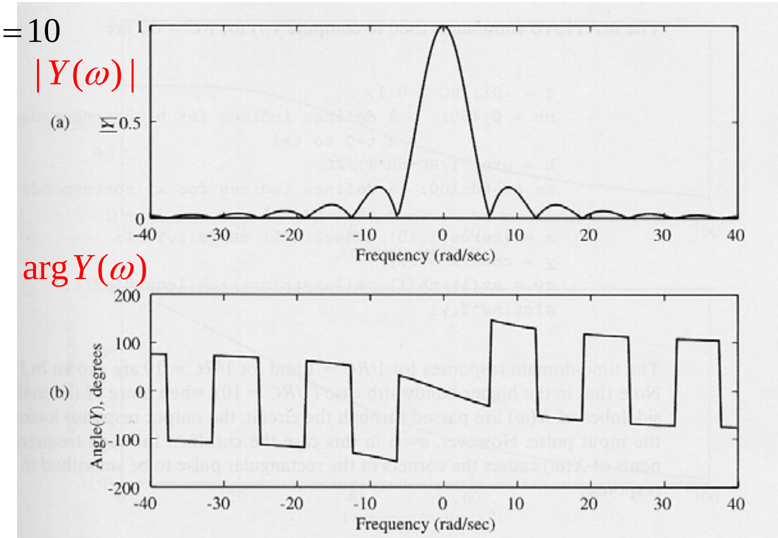


Figure 5.11 (a) Amplitude and (b) phase spectra of $y(t)$ when $1/RC = 10$.

Example: Response of an RC Circuit to a Rectangular Pulse – Cont'd

- The response of the system in the time domain can be found by computing the convolution

$$y(t) = h(t) * x(t)$$

where

$$h(t) = (1/RC)e^{-(1/RC)t}u(t)$$
$$x(t) = \text{rect}(t)$$

Example: Response of an RC Circuit to a Rectangular Pulse – Cont'd

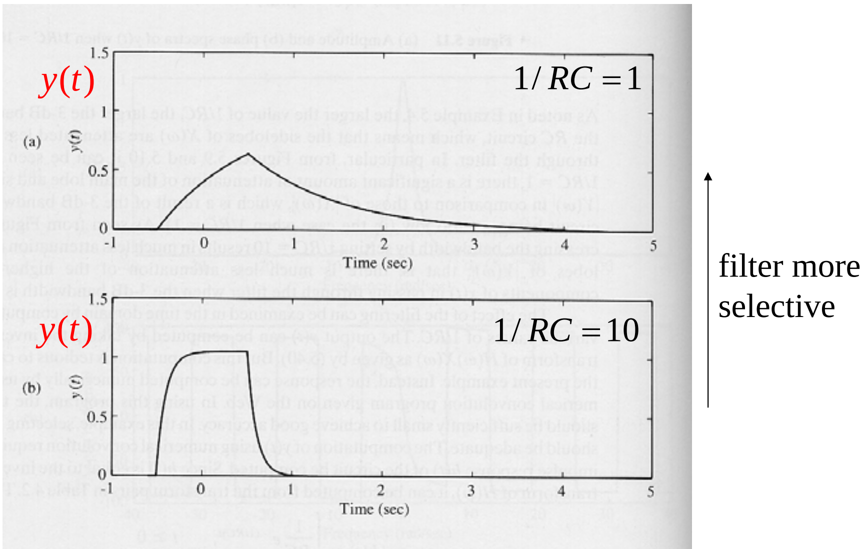
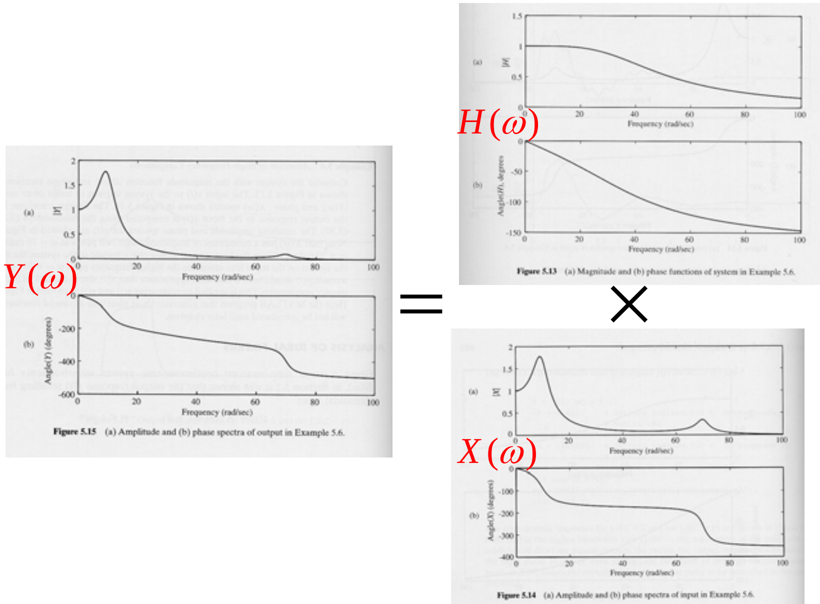


Figure 5.12 Output response when (a) $1/RC = 1$ and (b) $1/RC = 10$.

Example: Attenuation of High-Frequency Components



Example: Attenuation of High-Frequency Components

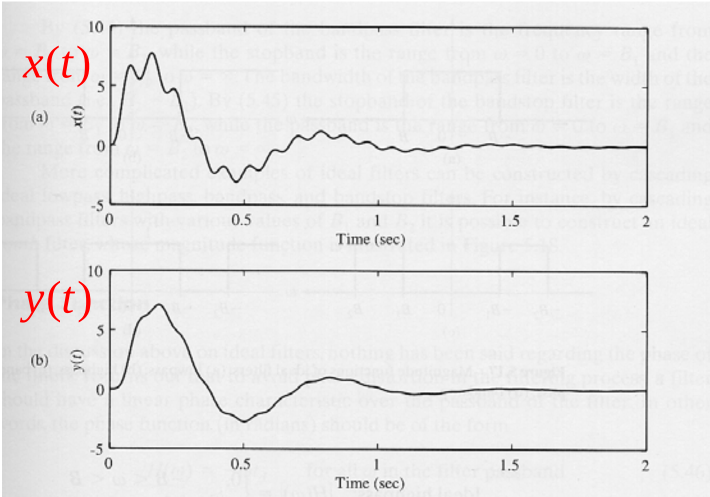


Figure 5.16 (a) Input and (b) resulting output in Example 5.6.

Filtering Signals

- The response of a CT, LTI system with frequency response $H(\omega)$ to a sinusoidal signal

$$x(t) = A\cos(\omega_0 t + \theta)$$

is

$$y(t) = A\,|H(\omega_0)|\cos\left(\omega_0 t + \theta + \arg H(\omega_0)\right)$$

- Filtering: if $|H(\omega_0)| = 0$ or $|H(\omega_0)| \approx 0$ then $y(t) = 0$ or $y(t) \approx 0, \ \forall t \in \mathbb{R}$

Four Basic Types of Filters

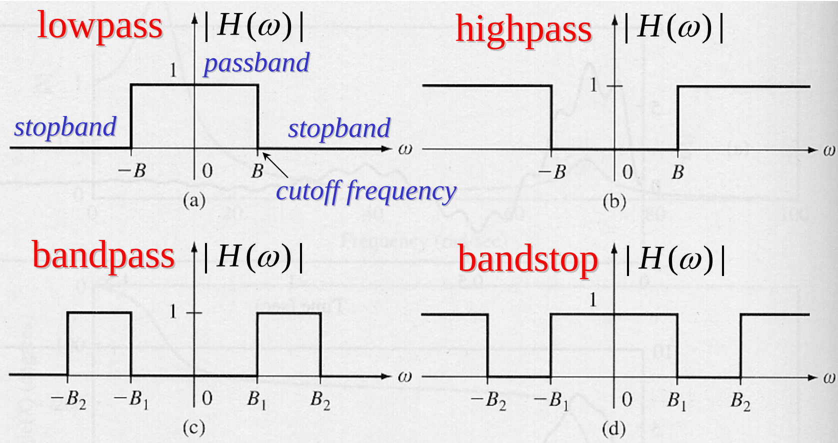


Figure 5.17 Magnitude functions of ideal filters: (a) lowpass; (b) highpass; (c) bandpass; (d) bandstop.

(many more details about filter design in ECE 464/564 and ECE 567)

Phase Function

- Filters are usually designed based on specifications on the magnitude response $|H(\omega)|$
- The phase response $\arg H(\omega)$ has to be taken into account too in order to prevent signal distortion as the signal goes through the system
- If the filter has linear phase in its passband(s), then there is no distortion

Linear-Phase Filters

- A filter $H(\omega)$ is said to have linear phase if

$$\arg H(\omega) = -\omega t_d, \quad \forall\, \omega \in \text{passband}$$

- If ω_0 is in passband of a linear phase filter, its response to

$$x(t) = A\cos(\omega_0 t)$$

is

$$y(t) = A\,|\,H(\omega_0)\,|\cos(\omega_0 t - \omega_0 t_d) =$$
$$= A\,|\,H(\omega_0)\,|\cos(\omega_0 (t - t_d))$$

Ideal Linear-Phase Lowpass

- The frequency response of an ideal lowpass filter is defined by

$$H(\omega) = \begin{cases} e^{-j\omega t_d}, & \omega \in [-B, B] \\ 0, & \omega \notin [-B, B] \end{cases}$$

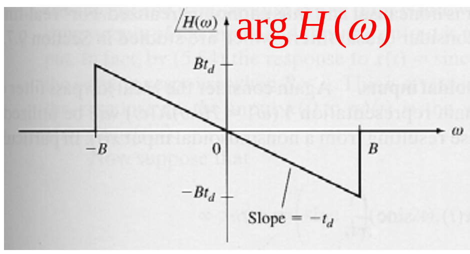


Figure 5.19 Phase function of ideal lowpass filter defined by (5.48).

Ideal Linear-Phase Lowpass – Cont'd

- $H(\omega)$ can be written as

$$H(\omega) = \text{rect}\left(\frac{\omega}{2B}\right)e^{-j\omega t_d}$$

whose inverse Fourier transform is

$$h(t) = \frac{B}{\pi}\text{sinc}\left(\frac{\text{B}}{\pi}(t - t_d)\right)$$

Ideal Linear-Phase Lowpass – Cont'd

$$h(t) = \frac{B}{\pi}\,\mathrm{sinc}\left(\frac{B}{\pi}(t - t_d)\right)$$

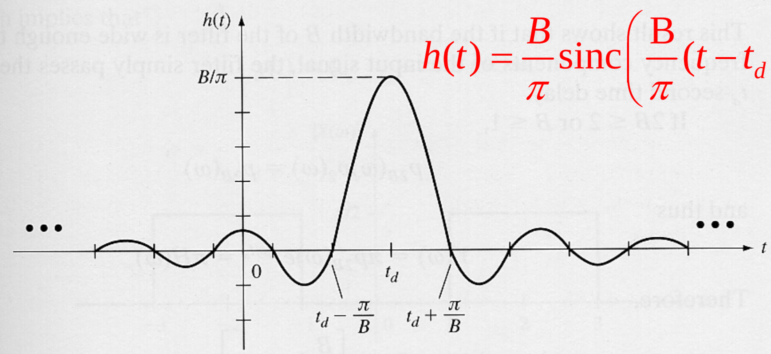
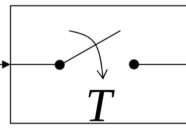


Figure 5.20 Impulse response of ideal linear-phase lowpass filter.

Notice: the filter is noncausal since $h(t)$ is not zero for $t < 0$

Ideal Sampling

- Consider the ideal sampler:

$x(t)$ →  → $x[n] = x(t)\big|_{t=nT} = x(nT)$

$t \in \mathbb{R}$ $\qquad$ $n \in \mathbb{Z}$

- It is convenient to express the sampled signal $x(nT)$ as $x(t)p(t)$ where

$$p(t) = \sum_{n \in \mathbb{Z}} \delta(t - nT)$$

Ideal Sampling – Cont'd

- Thus, the sampled waveform $x(t)\,p(t)$ is

$$x(t)\,p(t) = \sum_{n\in\mathbb{Z}} x(t)\delta(t-nT) = \sum_{n\in\mathbb{Z}} x(nT)\delta(t-nT)$$

- $x(t)\,p(t)$ is an impulse train whose weights (areas) are the sample values $x(nT)$ of the original signal $x(t)$

Ideal Sampling – Cont'd

- Since $p(t)$ is periodic with period T, it can be represented by its Fourier series

$$p(t) = \sum_{k \in \mathbb{Z}} c_k e^{jk\omega_s t}, \quad \omega_s = \frac{2\pi}{T}$$

sampling frequency (rad/sec)

where
$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} p(t) e^{-jk\omega_s t} dt, \quad k \in \mathbb{Z}$$
$$= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T}$$

Ideal Sampling – Cont'd

- Therefore $$p(t)=\sum_{k\in\mathbb{Z}}\frac{1}{T}e^{jk\omega_s t}$$

  and

$$x_s(t)=x(t)\,p(t)=\sum_{k\in\mathbb{Z}}\frac{1}{T}x(t)e^{jk\omega_s t}=\frac{1}{T}\sum_{k\in\mathbb{Z}}x(t)e^{jk\omega_s t}$$

  whose Fourier transform is

$$X_s(\omega)=\frac{1}{T}\sum_{k\in\mathbb{Z}}X(\omega-k\omega_s)$$

Ideal Sampling – Cont'd

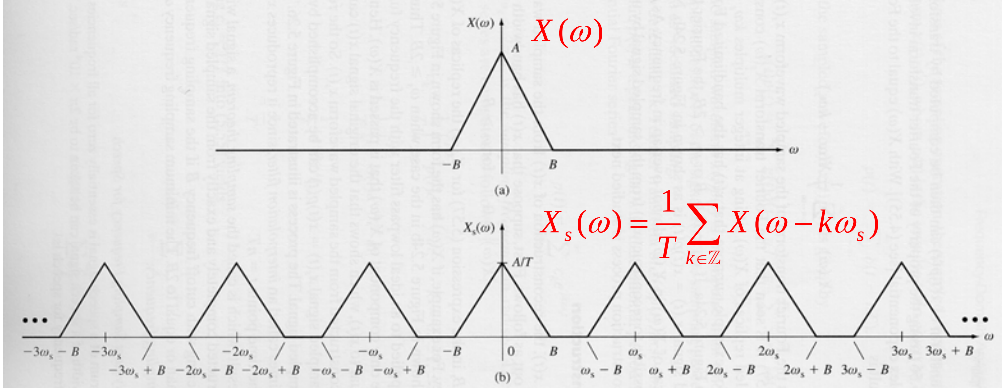


Figure 5.24 Fourier transform of (a) $x(t)$ and (b) $x_s(t) = x(t)p(t)$.

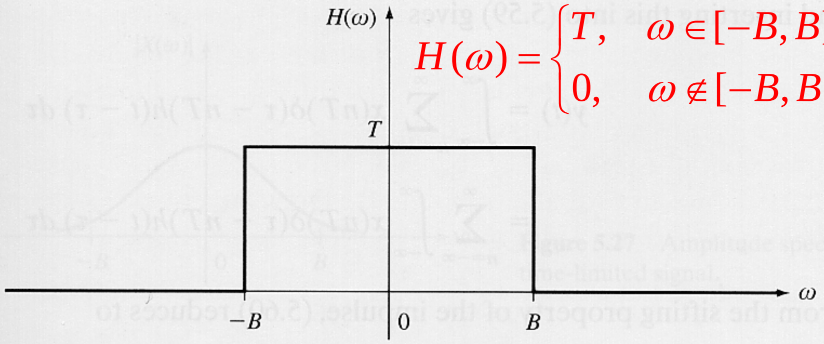
Signal Reconstruction

- Suppose that the signal $x(t)$ is bandlimited with bandwidth B, i.e., $|X(\omega)| = 0$, for $|\omega| > B$
- Then, if $\omega_s \geq 2B$, the replicas of $X(\omega)$ in

$$X_s(\omega) = \frac{1}{T}\sum_{k\in\mathbb{Z}} X(\omega - k\omega_s)$$

do not overlap and $X(\omega)$ can be recovered by applying an ideal lowpass filter to $X_s(\omega)$ (interpolation filter)

Interpolation Filter for Signal Reconstruction



$$H(\omega) = \begin{cases} T, & \omega \in [-B, B] \\ 0, & \omega \notin [-B, B] \end{cases}$$

Figure 5.25 Frequency response function of ideal lowpass filter with bandwidth *B*.

Interpolation Formula

- The impulse response $h(t)$ of the interpolation filter is

$$h(t) = \frac{BT}{\pi}\operatorname{sinc}\left(\frac{\mathrm{B}}{\pi}t\right)$$

and the output $y(t)$ of the interpolation filter is given by

$$y(t) = h(t) * x_s(t)$$

Interpolation Formula – Cont'd

- But

$$x_s(t) = x(t)\,p(t) = \sum_{n\in\mathbb{Z}} x(nT)\delta(t-nT)$$

whence

$$y(t) = h(t) * x_s(t) = \sum_{n\in\mathbb{Z}} x(nT)h(t-nT) =$$

$$= \frac{BT}{\pi}\sum_{n\in\mathbb{Z}} x(nT)\,\mathrm{sinc}\left(\frac{B}{\pi}(t-nT)\right)$$

- Moreover, $y(t) = x(t)$

Shannon's Sampling Theorem

- A CT bandlimited signal $x(t)$ with frequencies no higher than B can be reconstructed from its samples $x[n] = x(nT)$ if the samples are taken at a rate

$$\omega_s = 2\pi / T \geq 2B$$

- The reconstruction of $x(t)$ from its samples $x[n] = x(nT)$ is provided by the interpolation formula

$$x(t) = \frac{BT}{\pi} \sum_{n \in \mathbb{Z}} x(nT) \operatorname{sinc}\left(\frac{B}{\pi}(t - nT) \right)$$

Nyquist Rate

- The minimum sampling rate $\omega_s = 2\pi / T = 2B$ is called the Nyquist rate

- Question: Why do CD's adopt a sampling rate of 44.1 *kHz*?
- Answer: Since the highest frequency perceived by humans is about 20 *kHz*, 44.1 *kHz* is slightly more than twice this upper bound

Aliasing

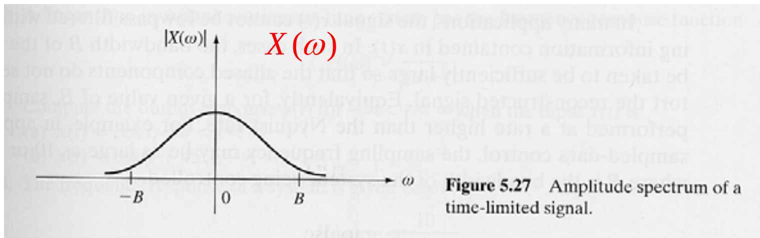


Figure 5.27 Amplitude spectrum of a time-limited signal.

$$X_s(\omega) = \frac{1}{T}\sum_{k\in\mathbb{Z}} X(\omega - k\omega_s)$$

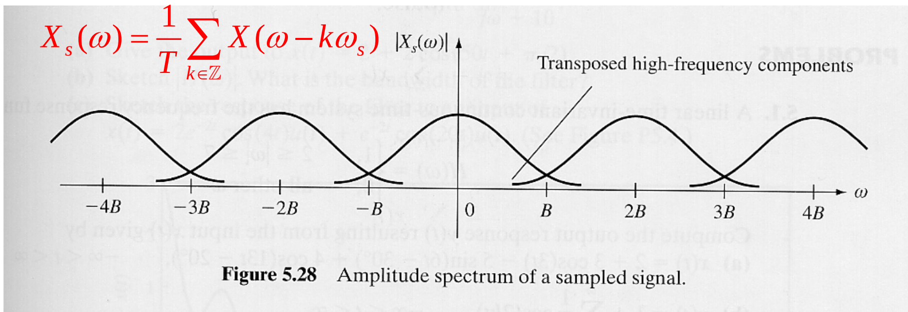


Figure 5.28 Amplitude spectrum of a sampled signal.

Aliasing –Cont'd

- Because of aliasing, it is not possible to reconstruct $x(t)$ exactly by lowpass filtering the sampled signal $x_s(t) = x(t)\,p(t)$
- Aliasing results in a distorted version of the original signal $x(t)$
- It can be eliminated (theoretically) by lowpass filtering $x(t)$ before sampling it so that $|X(\omega)| = 0$ for $|\omega| \geq B$