Chapter 4
The Fourier Series and Fourier Transform

Representation of Signals in Terms of Frequency Components

- Consider the CT signal defined by

$$x(t)=\sum_{k=1}^{N} A_k \cos(\omega_k t+\theta_k), \quad t\in\mathbb{R}$$

- The frequencies `present in the signal' are the frequency ω_k of the component sinusoids
- The signal $x(t)$ is completely characterized by the set of frequencies ω_k , the set of amplitudes A_k , and the set of phases θ_k

Example: Sum of Sinusoids

- Consider the CT signal given by

$$x(t)=A_1\cos(t)+A_2\cos(4t+\pi/3)+A_3\cos(8t+\pi/2),$$
$$t\in\mathbb{R}$$

- The signal has only three frequency components at 1,4, and 8 *rad/sec*, amplitudes A_1, A_2, A_3 and phases $0, \pi/3, \pi/2$
- The shape of the signal $x(t)$ depends on the relative magnitudes of the frequency components, specified in terms of the amplitudes A_1, A_2, A_3

Example: Sum of Sinusoids –Cont'd

$$\begin{cases} A_1=0.5 \\ A_2=1 \\ A_3=0 \end{cases}$$

$$\begin{cases} A_1=1 \\ A_2=0.5 \\ A_3=0 \end{cases}$$

$$\begin{cases} A_1=1 \\ A_2=1 \\ A_3=0 \end{cases}$$

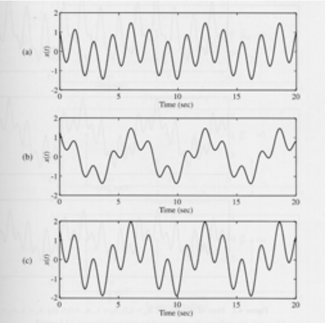

Figure 4.1 Plots of $x(t)$ for (a) $A_1 = 0.5, A_2 = 1, A_3 = 0$; (b) $A_1 = 1, A_2 = 0.5, A_3 = 0$; and (c) $A_1 = 1, A_2 = 1, A_3 = 0$.

Example: Sum of Sinusoids –Cont'd

$$\begin{cases} A_1=0.5 \\ A_2=1 \\ A_3=0.5 \end{cases}$$

$$\begin{cases} A_1=1 \\ A_2=0.5 \\ A_3=0.5 \end{cases}$$

$$\begin{cases} A_1=1 \\ A_2=1 \\ A_3=1 \end{cases}$$

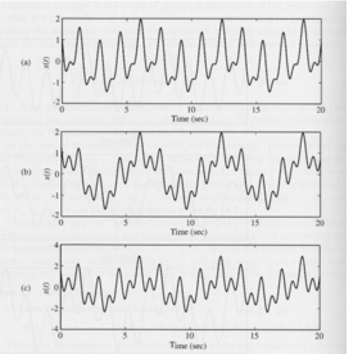

Figure 4.2 Plots of $x(t)$ for (a) $A_1 = 0.5, A_2 = 1, A_3 = 0.5$; (b) $A_1 = 1, A_2 = 0.5,$

Amplitude Spectrum

- Plot of the amplitudes A_k of the sinusoids making up $x(t)$ vs. ω
- Example:

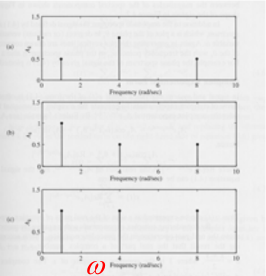

Figure 4.3 Amplitude spectra of the versions of $x(t)$ plotted in Figure 4.2.

Phase Spectrum

- Plot of the phases θ_k of the sinusoids making up $x(t)$ vs. ω
- Example:

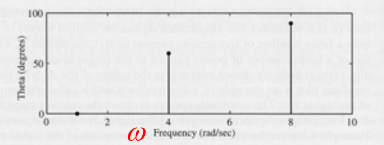

Figure 4.4 Phase spectrum of signal $x(t)$ defined by (4.2).

Complex Exponential Form

- Euler formula: $e^{j\alpha} = \cos(\alpha) + j\sin(\alpha)$
- Thus

$$A_k \cos(\omega_k t + \theta_k) = \underset{\text{real part}}{\Re}\left[A_k e^{j(\omega_k t + \theta_k)} \right]$$

whence

$$x(t) = \sum_{k=1}^{N} \Re\left[A_k e^{j(\omega_k t + \theta_k)} \right], \quad t \in \mathbb{R}$$

Complex Exponential Form – Cont'd

- And, recalling that $\Re(z) = (z + z^*)/2$ where $z = a + jb$, we can also write

$$x(t) = \sum_{k=1}^{N} \frac{1}{2}\left[A_k e^{j(\omega_k t + \theta_k)} + A_k e^{-j(\omega_k t + \theta_k)} \right], \quad t \in \mathbb{R}$$

- This signal contains both positive and negative frequencies
- The *negative frequencies* $-\omega_k$ stem from writing the *cosine* in terms of complex exponentials and have no physical meaning

Complex Exponential Form – Cont'd

- By defining

$$c_k = \frac{A_k}{2} e^{j\theta_k} \qquad c_{-k} = \frac{A_k}{2} e^{-j\theta_k}$$

it is also

$$x(t) = \sum_{k=1}^{N} \left[c_k e^{j\omega_k t} + c_{-k} e^{-j\omega_k t} \right] = \underbrace{\sum_{\substack{k=-N \\ k \neq 0}}^{N} c_k e^{j\omega_k t}}_{\substack{\text{complex exponential form} \\ \text{of the signal } x(t)}}, \quad t \in \mathbb{R}$$

Line Spectra

- The *amplitude spectrum* of $x(t)$ is defined as the plot of the magnitudes $|c_k|$ versus ω
- The *phase spectrum* of $x(t)$ is defined as the plot of the angles $\angle c_k = \arg(c_k)$ versus ω
- This results in *line spectra* which are defined for both positive and negative frequencies
- Notice: for $k = 1, 2, \ldots$

$$|c_k| = |c_{-k}| \qquad \angle c_k = -\angle c_{-k}$$

$$\arg(c_k) = -\arg(c_{-k})$$

Example: Line Spectra

$$x(t) = \cos(t) + 0.5\cos(4t + \pi/3) + \cos(8t + \pi/2)$$

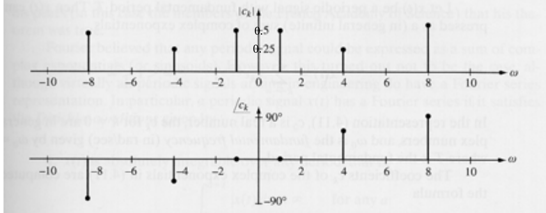

Figure 4.5 Line spectra for the signal in Example 4.2.

Fourier Series Representation of Periodic Signals

- Let $x(t)$ be a CT periodic signal with period T, *i.e.*, $x(t+T) = x(t), \quad \forall t \in R$
- Example: the rectangular pulse train

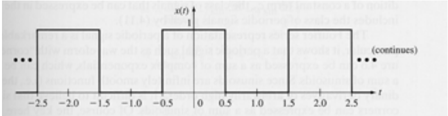

Figure 4.6 Periodic signal with fundamental period $T = 2$.

The Fourier Series

- Then, $x(t)$ can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad t \in \mathbb{R}$$

where $\omega_0 = 2\pi / T$ is the *fundamental frequency* (*rad/sec*) of the signal and

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_o t} dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

c_0 is called the *constant or dc component* of $x(t)$

The Fourier Series – Cont'd

- The frequencies $k\omega_0$ present in $x(t)$ are integer multiples of the fundamental frequency ω_0
- Notice that, if the dc term c_0 is added to

$$x(t) = \sum_{\substack{k=-N \\ k \neq 0}}^{N} c_k e^{j\omega_k t}$$

and we set $N = \infty$, the Fourier series is a special case of the above equation where all the frequencies are integer multiples of ω_0

Dirichlet Conditions

- A periodic signal $x(t)$, has a Fourier series if it satisfies the following conditions:

1. $x(t)$ is absolutely integrable over any period, namely

$$\int_{a}^{a+T} |x(t)| \, dt < \infty, \quad \forall a \in \mathbb{R}$$

2. $x(t)$ has only a finite number of maxima and minima over any period

3. $x(t)$ has only a finite number of discontinuities over any period

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Example: The Rectangular Pulse Train

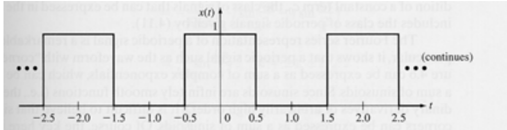

Figure 4.6 Periodic signal with fundamental period $T = 2$.

- From figure, $T = 2$ whence $\omega_0 = 2\pi / 2 = \pi$
- Clearly $x(t)$ satisfies the Dirichlet conditions and thus has a Fourier series representation

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Example: The Rectangular Pulse Train – Cont'd

$$x(t) = \frac{1}{2} + \sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} \frac{1}{k\pi} (-1)^{|(k-1)/2|} e^{jk\pi t}, \quad t \in \mathbb{R}$$

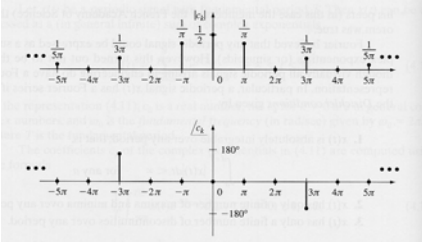


Figure 4.7 Line spectra for the rectangular pulse train.

Trigonometric Fourier Series

- By using Euler's formula, we can rewrite

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad t \in \mathbb{R}$$

as

$$x(t) = \underset{\text{dc component}}{c_0} + \sum_{k=1}^{\infty} \underbrace{2\,|c_k|\cos(k\omega_0 t + \angle c_k)}_{k\text{-th harmonic}}, \quad t \in \mathbb{R}$$

- This expression is called the trigonometric Fourier series of $x(t)$

Example: Trigonometric Fourier Series of the Rectangular Pulse Train

- The expression

$$x(t) = \frac{1}{2} + \sum_{\substack{k=-\infty \\ k\ \text{odd}}}^{\infty} \frac{1}{k\pi}(-1)^{|(k-1)/2|} e^{jk\pi t}, \quad t \in \mathbb{R}$$

can be rewritten as

$$x(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k\ \text{odd}}}^{\infty} \frac{2}{k\pi}\cos\left(k\pi t + \left[(-1)^{(k-1)/2} - 1\right]\frac{\pi}{2}\right), \quad t \in \mathbb{R}$$

Gibbs Phenomenon

- Given an odd positive integer N, define the N-th partial sum of the previous series

$$x_N(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k\ \text{odd}}}^{N} \frac{2}{k\pi}\cos\left(k\pi t + \left[(-1)^{(k-1)/2} - 1\right]\frac{\pi}{2}\right), \quad t \in \mathbb{R}$$

- According to Fourier's theorem, it should be

$$\lim_{N \to \infty} |x_N(t) - x(t)| = 0$$

Gibbs Phenomenon – Cont'd

$x_3(t)$

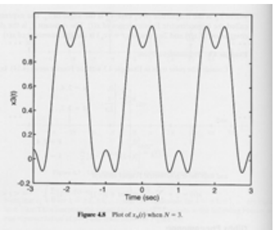


Figure 4.8 Plot of $x_3(t)$ when $N = 3$.

$x_9(t)$

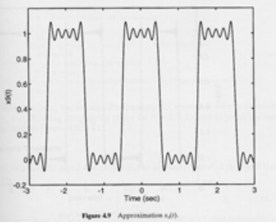


Figure 4.9 Approximation $x_9(t)$.

Gibbs Phenomenon – Cont'd

$x_{21}(t)$

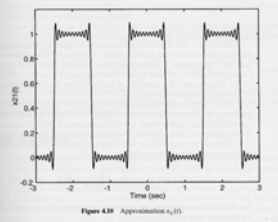


Figure 4.10 Approximation $x_{21}(t)$.

$x_{45}(t)$

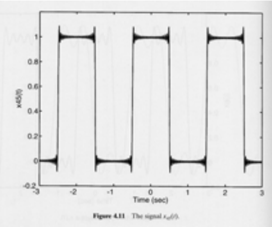


Figure 4.11 The signal $x_{45}(t)$.

overshoot: about 9 % of the signal magnitude
(present even if $N \to \infty$)

Parseval's Theorem

- Let $x(t)$ be a periodic signal with period T
- The *average power* P of the signal is defined as

$$P = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt$$

- Expressing the signal as $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad t \in \mathbb{R}$ it is also

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Fourier Transform

- We have seen that periodic signals can be represented with the Fourier series
- Can aperiodic signals be analyzed in terms of frequency components?
- Yes, and the Fourier transform provides the tool for this analysis
- The major difference w.r.t. the line spectra of periodic signals is that the spectra of aperiodic signals are defined for all real values of the frequency variable ω not just for a discrete set of values

Frequency Content of the Rectangular Pulse

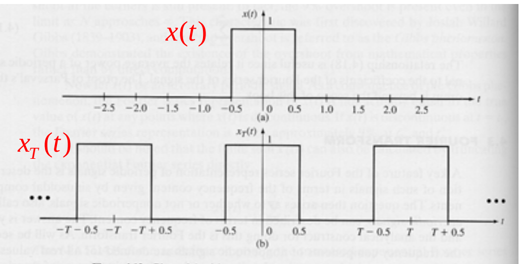


Figure 4.12 Plots of the (a) one-second rectangular pulse and (b) pulse train.

$$x(t) = \lim_{T\to\infty} x_T(t)$$

Frequency Content of the Rectangular Pulse – Cont'd

- Since $x_T(t)$ is periodic with period T, we can write

$$x_T(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad t \in \mathbb{R}$$

where

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_o t}\,dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

Frequency Content of the Rectangular Pulse – Cont'd

- What happens to the frequency components of $x_T(t)$ as $T \to \infty$?
- For $k = 0$

$$c_0 = \frac{1}{T}$$

- For $k \neq 0$

$$c_k = \frac{2}{k\omega_0 T}\sin\left(\frac{k\omega_0}{2}\right) \underset{\substack{\uparrow \\ \omega_0 = 2\pi/T}}{=} \frac{1}{k\pi}\sin\left(\frac{k\omega_0}{2}\right), \quad k = \pm 1, \pm 2, \ldots$$

Frequency Content of the Rectangular Pulse – Cont'd

plots of $T\,|\,c_k\,|$ vs. $\omega = k\omega_0$ for $T = 2,5,10$

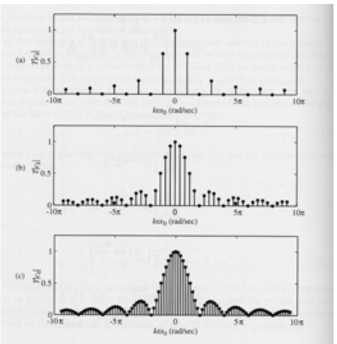


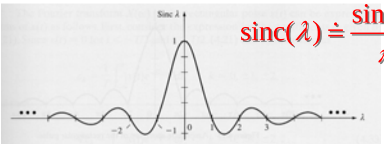
Figure 4.13 Plot of scaled spectrum of $x_T(t)$ for (a) $T = 2$, (b) $T = 5$, and (c) $T = 10$.

Frequency Content of the Rectangular Pulse – Cont'd

- It can be easily shown that

$$\lim_{T\to\infty} Tc_k = \operatorname{sinc}\left(\frac{\omega}{2\pi}\right), \quad \omega \in \mathbb{R}$$

where



$$\operatorname{sinc}(\lambda) \doteq \frac{\sin(\pi\lambda)}{\pi\lambda}$$

Figure 4.14 Plot of sinc λ.

Fourier Transform of the Rectangular Pulse

- The Fourier transform of the rectangular pulse $x(t)$ is defined to be the limit of Tc_k as $T \to \infty$, i.e.,

$$X(\omega) = \lim_{T\to\infty} Tc_k = \operatorname{sinc}\left(\frac{\omega}{2\pi}\right), \quad \omega \in \mathbb{R}$$

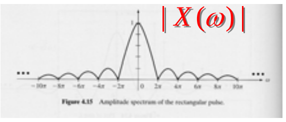


Figure 4.15 Amplitude spectrum of the rectangular pulse.

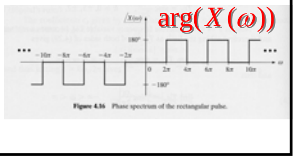


Figure 4.16 Phase spectrum of the rectangular pulse.

Fourier Transform of the Rectangular Pulse – Cont'd

- The Fourier transform $X(\omega)$ of the rectangular pulse $x(t)$ can be expressed in terms of $x(t)$ as follows:

$$c_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_o t}dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

$x(t) = 0$ for $t < -T/2$ and $t > T/2$

whence

$$Tc_k = \int_{-\infty}^{\infty} x(t)e^{-jk\omega_o t}dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

Fourier Transform of the Rectangular Pulse – Cont'd

- Now, by definition $X(\omega) = \lim_{T\to\infty} Tc_k$ and, since $k\omega_0 \to \omega$ as $T \to \infty$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \quad \omega \in \mathbb{R}$$

- The *inverse Fourier transform* of $X(\omega)$ is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega, \quad t \in \mathbb{R}$$

The Fourier Transform in the General Case

- Given a signal $x(t)$, its *Fourier transform* $X(\omega)$ is defined as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \quad \omega \in \mathbb{R}$$

- A signal $x(t)$ is said to have a *Fourier transform in the ordinary sense* if the above integral converges

The Fourier Transform in the General Case – Cont'd

- The integral does converge if
  1. the signal $x(t)$ is *"well-behaved"*
  2. and $x(t)$ is *absolutely integrable*, namely,

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

- Note: *well behaved* means that the signal has a finite number of discontinuities, maxima, and minima within any finite time interval

Example: The DC or Constant Signal

- Consider the signal $x(t) = 1, \quad t \in \mathbb{R}$
- Clearly $x(t)$ does not satisfy the first requirement since

$$\int_{-\infty}^{\infty} |x(t)|\, dt = \int_{-\infty}^{\infty} dt = \infty$$

- Therefore, the constant signal does not have a *Fourier transform in the ordinary sense*
- Later on, we'll see that it has however a *Fourier transform in a generalized sense*

Example: The Exponential Signal

- Consider the signal $x(T) = e^{-bt}u(t), \quad b \in \mathbb{R}$
- Its Fourier transform is given by

$$X(\omega) = \int_{-\infty}^{\infty} e^{-bt}u(t)e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-(b+j\omega)t}dt = -\frac{1}{b+j\omega}\left[e^{-(b+j\omega)t}\right]_{t=0}^{t=\infty}$$

Example: The Exponential Signal – Cont'd

- If $b < 0$, $X(\omega)$ does not exist
- If $b = 0$, $x(t) = u(t)$ and $X(\omega)$ does not exist either in the ordinary sense
- If $b > 0$, it is

$$X(\omega) = \frac{1}{b+j\omega}$$

amplitude spectrum

$$|X(\omega)| = \frac{1}{\sqrt{b^2+\omega^2}}$$

phase spectrum

$$\arg(X(\omega)) = -\arctan\left(\frac{\omega}{b}\right)$$

Example: Amplitude and Phase Spectra of the Exponential Signal

$x(t) = e^{-10t}u(t)$

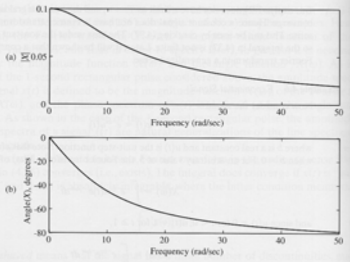


Figure 4.17 Plots of the (a) amplitude and (b) phase spectra of $x(t) = \exp(-10t)u(t)$.

Rectangular Form of the Fourier Transform

- Consider

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \quad \omega \in \mathbb{R}$$

- Since $X(\omega)$ in general is a complex function, by using Euler's formula

$$X(\omega) = \underbrace{\int_{-\infty}^{\infty} x(t)\cos(\omega t)dt}_{R(\omega)} + j\underbrace{\left(-\int_{-\infty}^{\infty} x(t)\sin(\omega t)dt\right)}_{I(\omega)}$$

$$X(\omega) = R(\omega) + jI(\omega)$$

Polar Form of the Fourier Transform

- $X(\omega) = R(\omega) + jI(\omega)$ can be expressed in a polar form as

$$X(\omega) = |X(\omega)|\exp(j\arg(X(\omega)))$$

where

$$|X(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)}$$

$$\arg(X(\omega)) = \arctan\left(\frac{I(\omega)}{R(\omega)}\right)$$

Fourier Transform of Real-Valued Signals

- If $x(t)$ is real-valued, it is

$$X(-\omega) = X^*(\omega)$$

Hermitian symmetry

- Moreover

$$X^*(\omega) = |X(\omega)|\exp(-j\arg(X(\omega)))$$

whence

$$|X(-\omega)| = |X(\omega)| \quad \text{and}$$

$$\arg(X(-\omega)) = -\arg(X(\omega))$$

Fourier Transforms of Signals with Even or Odd Symmetry

- Even signal: $x(t) = x(-t)$

$$X(\omega) = 2\int_0^{\infty} x(t)\cos(\omega t)dt$$

- Odd signal: $x(t) = -x(-t)$

$$X(\omega) = -j2\int_0^{\infty} x(t)\sin(\omega t)dt$$

Example: Fourier Transform of the Rectangular Pulse

- Consider the even signal

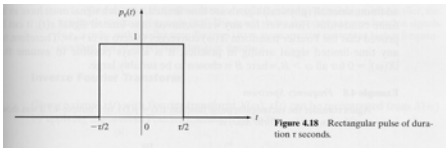


Figure 4.18 Rectangular pulse of duration τ seconds.

- It is

$$X(\omega) = 2\int_0^{\tau/2} (1)\cos(\omega t)dt = \frac{2}{\omega}\left[\sin(\omega t)\right]_{t=0}^{t=\tau/2} = \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right)$$

$$= \tau\,\text{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

Example: Fourier Transform of the Rectangular Pulse – Cont'd

$$X(\omega) = \tau\,\text{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

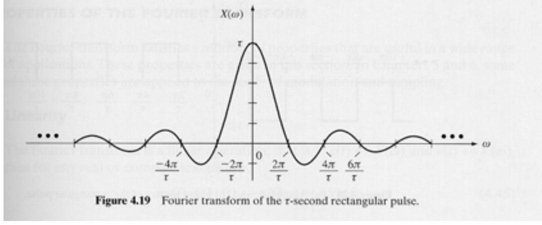


Figure 4.19 Fourier transform of the τ-second rectangular pulse.

Example: Fourier Transform of the Rectangular Pulse – Cont'd

amplitude spectrum

phase spectrum

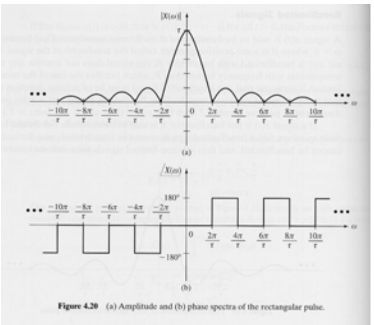


Figure 4.20 (a) Amplitude and (b) phase spectra of the rectangular pulse.

Bandlimited Signals

- A signal $x(t)$ is said to be *bandlimited* if its Fourier transform $X(\omega)$ is zero for all $\omega > B$ where B is some positive number, called the *bandwidth of the signal*
- It turns out that any bandlimited signal must have an infinite duration in time, i.e., bandlimited signals cannot be time limited

Bandlimited Signals – Cont'd

- If a signal $x(t)$ is not bandlimited, it is said to have *infinite bandwidth* or an *infinite spectrum*
- Time-limited signals cannot be bandlimited and thus all time-limited signals have infinite bandwidth
- However, for any well-behaved signal $x(t)$ it can be proven that $\lim_{\omega\to\infty} X(\omega) = 0$ whence it can be assumed that

$$|X(\omega)| \approx 0 \quad \forall \omega > B$$

B being a convenient large number

Inverse Fourier Transform

- Given a signal $x(t)$ with Fourier transform $X(\omega)$, $x(t)$ can be recomputed from $X(\omega)$ by applying the inverse Fourier transform given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega, \quad t \in \mathbb{R}$$

- Transform pair

$$x(t) \leftrightarrow X(\omega)$$

Properties of the Fourier Transform

$$x(t) \leftrightarrow X(\omega) \qquad y(t) \leftrightarrow Y(\omega)$$

- *Linearity:*

$$\alpha x(t) + \beta y(t) \leftrightarrow \alpha X(\omega) + \beta Y(\omega)$$

- *Left or Right Shift in Time:*

$$x(t - t_0) \leftrightarrow X(\omega)e^{-j\omega t_0}$$

- *Time Scaling:*

$$x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right)$$

Properties of the Fourier Transform

- *Time Reversal:*

$$x(-t) \leftrightarrow X(-\omega)$$

- *Multiplication by a Power of t:*

$$t^n x(t) \leftrightarrow (j)^n \frac{d^n}{d\omega^n} X(\omega)$$

- *Multiplication by a Complex Exponential:*

$$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

Properties of the Fourier Transform

- *Multiplication by a Sinusoid (Modulation):*

$$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}\left[X(\omega + \omega_0) - X(\omega - \omega_0)\right]$$

$$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[X(\omega + \omega_0) + X(\omega - \omega_0)\right]$$

- *Differentiation in the Time Domain:*

$$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega)$$

Properties of the Fourier Transform

- *Integration in the Time Domain:*

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$$

- *Convolution in the Time Domain:*

$$x(t) * y(t) \leftrightarrow X(\omega)Y(\omega)$$

- *Multiplication in the Time Domain:*

$$x(t)y(t) \leftrightarrow X(\omega) * Y(\omega)$$

Properties of the Fourier Transform

- *Parseval's Theorem:*

$$\int_{\mathbb{R}} x(t)y(t)dt \leftrightarrow \frac{1}{2\pi}\int_{\mathbb{R}} X^*(\omega)Y(\omega)d\omega$$

$$\text{if } y(t) = x(t) \quad \int_{\mathbb{R}} |x(t)|^2\, dt \leftrightarrow \frac{1}{2\pi}\int_{\mathbb{R}} |X(\omega)|^2 d\omega$$

- *Duality:*

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

Properties of the Fourier Transform - Summary

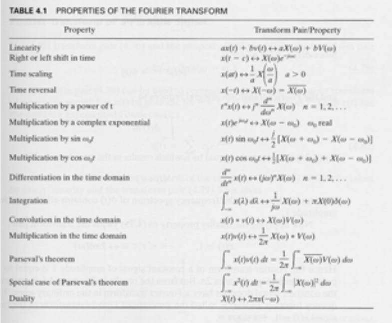


TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Property	Transform Pair/Property
Linearity	$ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$
Right or left shift in time	$x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$
Time scaling	$x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right) \quad a > 0$
Time reversal	$x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$
Multiplication by a power of t	$t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$
Multiplication by a complex exponential	$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real
Multiplication by sin $\omega_0 t$	$x(t)\sin\omega_0 t \leftrightarrow \frac{j}{2}[X(\omega+\omega_0) - X(\omega-\omega_0)]$
Multiplication by cos $\omega_0 t$	$x(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}[X(\omega+\omega_0) + X(\omega-\omega_0)]$
Differentiation in the time domain	$\frac{d^n}{dt^n}x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$
Integration	$\int_{-\infty}^{t} x(\lambda)\,d\lambda \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$
Convolution in the time domain	$x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$
Multiplication in the time domain	$x(t)v(t) \leftrightarrow \frac{1}{2\pi}X(\omega) * V(\omega)$
Parseval's theorem	$\int_{-\infty}^{\infty} x(t)v(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)\,d\omega$
Special case of Parseval's theorem	$\int_{-\infty}^{\infty} x^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\,d\omega$
Duality	$X(t) \leftrightarrow 2\pi x(-\omega)$

Example: Linearity

$$x(t) = p_4(t) + p_2(t)$$

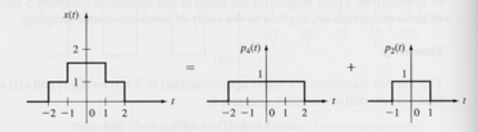


Figure 4.21 Signal in Example 4.9.

$$X(\omega) = 4\text{sinc}\left(\frac{2\omega}{\pi}\right) + 2\text{sinc}\left(\frac{\omega}{\pi}\right)$$

Example: Time Shift

$$x(t) = p_2(t-1)$$

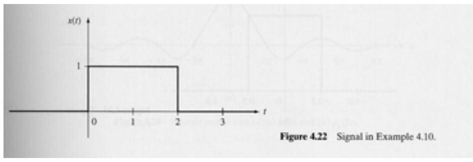
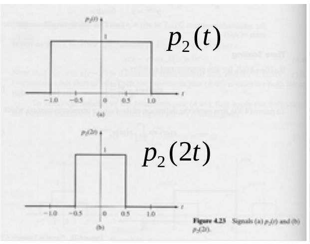


Figure 4.22 Signal in Example 4.10.

$$X(\omega) = 2\text{sinc}\left(\frac{\omega}{\pi}\right)e^{-j\omega}$$

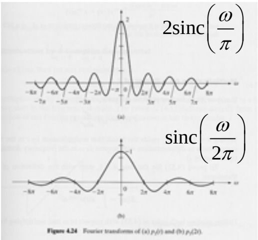
Example: Time Scaling



$p_2(t)$

$p_2(2t)$

Figure 4.23 Signals (a) $p_2(t)$ and (b) $p_2(2t)$.



$2\text{sinc}\left(\frac{\omega}{\pi}\right)$

$\text{sinc}\left(\frac{\omega}{2\pi}\right)$

Figure 4.24 Fourier transforms of (a) $p_2(t)$ and (b) $p_2(2t)$.

$a > 1$ time compression ↔ frequency expansion

$0 < a < 1$ time expansion ↔ frequency compression

Example: Multiplication in Time

$$x(t) = tp_2(t)$$

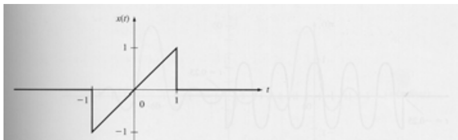


Figure 4.25 The signal $x(t) = tp_2(t)$.

$$X(\omega) = j\frac{d}{d\omega}\left(2\text{sinc}\left(\frac{\omega}{\pi}\right)\right) = j2\frac{d}{d\omega}\left(\frac{\sin\omega}{\omega}\right) = j2\frac{\omega\cos\omega - \sin\omega}{\omega^2}$$

Example: Multiplication in Time – Cont'd

$$X(\omega) = j2\frac{\omega\cos\omega - \sin\omega}{\omega^2}$$

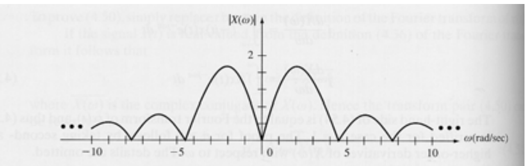


Figure 4.26 Amplitude spectrum of the signal in Figure 4.25.

Example: Multiplication by a Sinusoid

$$x(t) = p_\tau(t)\cos(\omega_0 t)$$ sinusoidal burst

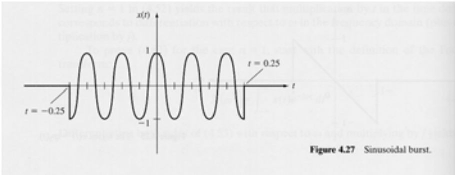
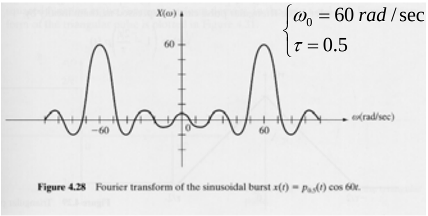


Figure 4.27 Sinusoidal burst.

$$X(\omega) = \frac{1}{2}\left[\tau\,\mathrm{sinc}\left(\frac{\tau(\omega+\omega_0)}{2\pi}\right) + \tau\,\mathrm{sinc}\left(\frac{\tau(\omega-\omega_0)}{2\pi}\right)\right]$$

Example: Multiplication by a Sinusoid – Cont'd

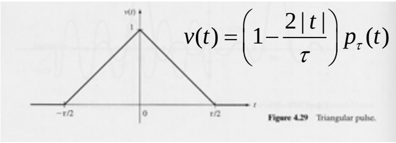
$$X(\omega) = \frac{1}{2}\left[\tau\,\mathrm{sinc}\left(\frac{\tau(\omega+\omega_0)}{2\pi}\right) + \tau\,\mathrm{sinc}\left(\frac{\tau(\omega-\omega_0)}{2\pi}\right)\right]$$



$$\begin{cases} \omega_0 = 60\ rad/\sec \\ \tau = 0.5 \end{cases}$$

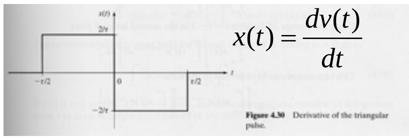
Figure 4.28 Fourier transform of the sinusoidal burst $x(t) = p_{0.5}(t)\cos 60t$.

Example: Integration in the Time Domain



$$v(t) = \left(1 - \frac{2|t|}{\tau}\right)p_\tau(t)$$

Figure 4.29 Triangular pulse.



$$x(t) = \frac{dv(t)}{dt}$$

Figure 4.30 Derivative of the triangular pulse.

Example: Integration in the Time Domain – Cont'd

- The Fourier transform of $x(t)$ can be easily found to be

$$X(\omega) = \left(\mathrm{sinc}\left(\frac{\tau\omega}{4\pi}\right)\right)\left(j2\sin\left(\frac{\tau\omega}{4}\right)\right)$$

- Now, by using the integration property, it is

$$V(\omega) = \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega) = \frac{\tau}{2}\mathrm{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$$

Example: Integration in the Time Domain – Cont'd

$$V(\omega) = \frac{\tau}{2}\mathrm{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$$

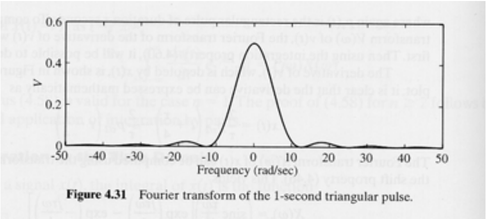


Figure 4.31 Fourier transform of the 1-second triangular pulse.

Generalized Fourier Transform

- Fourier transform of $\delta(t)$

$$\int_{\mathbb{R}} \delta(t)e^{-j\omega t}dt = 1 \quad \Rightarrow \quad \delta(t) \leftrightarrow 1$$

- Applying the duality property

$$x(t) = 1,\ t \in \mathbb{R} \leftrightarrow \underbrace{2\pi\delta(\omega)}_{}$$

generalized Fourier transform
of the constant signal $x(t) = 1, t \in \mathbb{R}$

Generalized Fourier Transform of Sinusoidal Signals

$$\cos(\omega_0 t) \leftrightarrow \pi\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right]$$

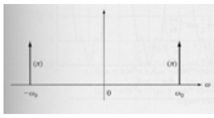


Figure 4.32 Fourier transform of $\cos \omega_0 t$.

$$\sin(\omega_0 t) \leftrightarrow j\pi\left[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right]$$

Fourier Transform of Periodic Signals

- Let $x(t)$ be a periodic signal with period T; as such, it can be represented with its Fourier transform

$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = 2\pi/T$$

- Since $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$, it is

$$X(\omega)=\sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega-k\omega_0)$$

Fourier Transform of the Unit-Step Function

- Since

$$u(t)=\int_{-\infty}^{t} \delta(\tau)d\tau$$

using the integration property, it is

$$u(t)=\int_{-\infty}^{t} \delta(\tau)d\tau \leftrightarrow \frac{1}{j\omega}+\pi\delta(\omega)$$

Common Fourier Transform Pairs

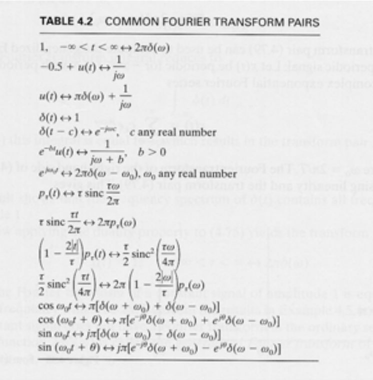


TABLE 4.2 COMMON FOURIER TRANSFORM PAIRS

$1, \ -\infty < t < \infty \leftrightarrow 2\pi\delta(\omega)$

$-0.5 + u(t) \leftrightarrow \dfrac{1}{j\omega}$

$u(t) \leftrightarrow \pi\delta(\omega) + \dfrac{1}{j\omega}$

$\delta(t) \leftrightarrow 1$

$\delta(t-c) \leftrightarrow e^{-j\omega c}$, c any real number

$e^{-bt}u(t) \leftrightarrow \dfrac{1}{j\omega + b}$, $b > 0$

$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$, ω_0 any real number

$p_\tau(t) \leftrightarrow \tau \operatorname{sinc} \dfrac{\tau\omega}{2\pi}$

$\tau \operatorname{sinc} \dfrac{\tau t}{2\pi} \leftrightarrow 2\pi p_\tau(\omega)$

$\left(1-\dfrac{2|t|}{\tau}\right)p_\tau(t) \leftrightarrow \dfrac{\tau}{2} \operatorname{sinc}^2\left(\dfrac{\tau\omega}{4\pi}\right)$

$\dfrac{\tau}{2} \operatorname{sinc}^2\left(\dfrac{\tau t}{4\pi}\right) \leftrightarrow 2\pi\left(1-\dfrac{2|\omega|}{\tau}\right)p_\tau(\omega)$

$\cos \omega_0 t \leftrightarrow \pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$

$\cos(\omega_0 t+\theta) \leftrightarrow \pi[e^{-j\theta}\delta(\omega+\omega_0)+e^{j\theta}\delta(\omega-\omega_0)]$

$\sin \omega_0 t \leftrightarrow j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$

$\sin(\omega_0 t+\theta) \leftrightarrow j\pi[e^{-j\theta}\delta(\omega+\omega_0)-e^{j\theta}\delta(\omega-\omega_0)]$