# Chapter 4
# The Fourier Series and Fourier Transform

## Representation of Signals in Terms of Frequency Components

- Consider the CT signal defined by

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_k t + \theta_k), \quad t \in \mathbb{R}$$

- The frequencies `present in the signal' are the frequency $\omega_k$ of the component sinusoids
- The signal $x(t)$ is completely characterized by the set of frequencies $\omega_k$ , the set of amplitudes $A_k$ , and the set of phases $\theta_k$

## Example: Sum of Sinusoids

- Consider the CT signal given by

$$x(t) = A_1 \cos(t) + A_2 \cos(4t + \pi/3) + A_3 \cos(8t + \pi/2),$$
$$t \in \mathbb{R}$$

- The signal has only three frequency components at 1,4, and 8 *rad/sec*, amplitudes $A_1, A_2, A_3$ and phases $0, \pi/3, \pi/2$
- The shape of the signal $x(t)$ depends on the relative magnitudes of the frequency components, specified in terms of the amplitudes $A_1, A_2, A_3$

## Example: Sum of Sinusoids –Cont'd

$$\begin{cases} A_1 = 0.5 \\ A_2 = 1 \\ A_3 = 0 \end{cases}$$

$$\begin{cases} A_1 = 1 \\ A_2 = 0.5 \\ A_3 = 0 \end{cases}$$

$$\begin{cases} A_1 = 1 \\ A_2 = 1 \\ A_3 = 0 \end{cases}$$

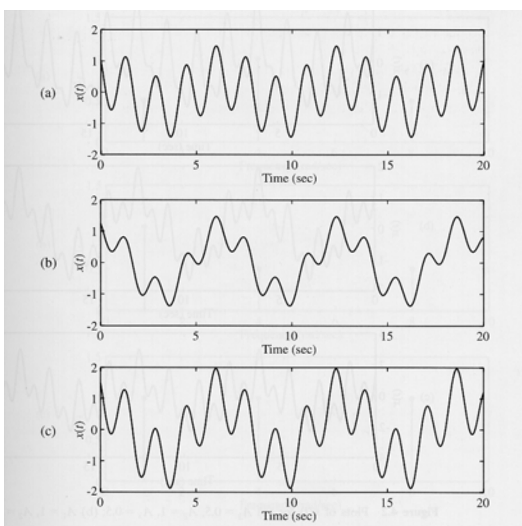


**Figure 4.1** Plots of $x(t)$ for (a) $A_1 = 0.5, A_2 = 1, A_3 = 0$; (b) $A_1 = 1, A_2 = 0.5, A_3 = 0$; and (c) $A_1 = 1, A_2 = 1, A_3 = 0$.

## Example: Sum of Sinusoids –Cont'd

$$\begin{cases} A_1 = 0.5 \\ A_2 = 1 \\ A_3 = 0.5 \end{cases}$$

$$\begin{cases} A_1 = 1 \\ A_2 = 0.5 \\ A_3 = 0.5 \end{cases}$$

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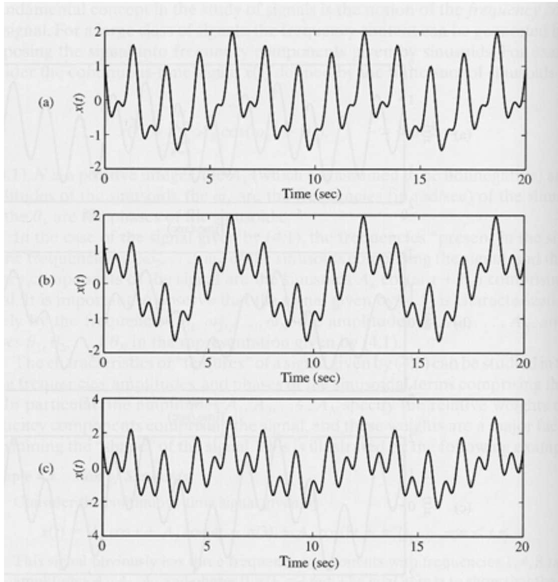



**Figure 4.2** Plots of $x(t)$ for (a) $A_1 = 0.5$, $A_2 = 1$, $A_3 = 0.5$; (b) $A_1 = 1$, $A_2 = 0.5$,

## Amplitude Spectrum

- Plot of the amplitudes $A_k$ of the sinusoids making up $x(t)$ vs. $\omega$
- Example:

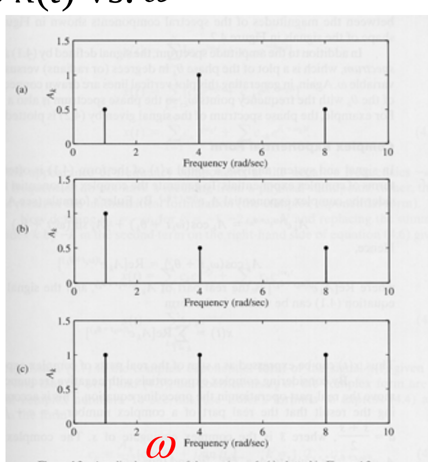



**Figure 4.3** Amplitude spectra of the versions of $x(t)$ plotted in Figure 4.2.

## Phase Spectrum

- Plot of the phases $\theta_k$ of the sinusoids making up $x(t)$ vs. $\omega$

- Example:

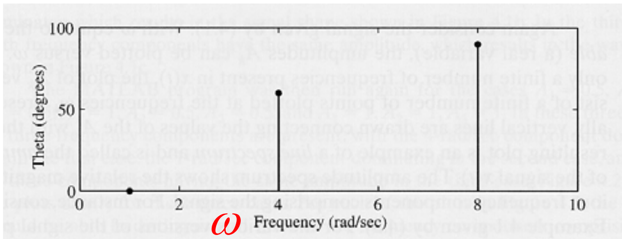



**Figure 4.4** Phase spectrum of signal $x(t)$ defined by (4.2).

## Complex Exponential Form

- Euler formula: $e^{j\alpha} = \cos(\alpha) + j\sin(\alpha)$
- Thus

$$A_k \cos(\omega_k t + \theta_k) = \underbrace{\Re}_{\text{real part}}\left[ A_k e^{j(\omega_k t + \theta_k)} \right]$$

whence

$$x(t) = \sum_{k=1}^{N} \Re\left[ A_k e^{j(\omega_k t + \theta_k)} \right], \quad t \in \mathbb{R}$$

## Complex Exponential Form – Cont'd

- And, recalling that $\Re(z) = (z + z^*)/2$ where $z = a + jb$, we can also write

$$x(t) = \sum_{k=1}^{N} \frac{1}{2}\left[ A_k e^{j(\omega_k t + \theta_k)} + A_k e^{-j(\omega_k t + \theta_k)} \right], \quad t \in \mathbb{R}$$

- This signal contains both positive and negative frequencies
- The *negative frequencies* $-\omega_k$ stem from writing the *cosine* in terms of complex exponentials and have no physical meaning

## Complex Exponential Form – Cont'd

- By defining

$$c_k = \frac{A_k}{2} e^{j\theta_k} \qquad c_{-k} = \frac{A_k}{2} e^{-j\theta_k}$$

it is also

$$x(t) = \sum_{k=1}^{N} \left[ c_k e^{j\omega_k t} + c_{-k} e^{-j\omega_k t} \right] = \underbrace{\sum_{\substack{k=-N \\ k \neq 0}}^{N} c_k e^{j\omega_k t}}_{\text{complex exponential form of the signal } x(t)}, \quad t \in \mathbb{R}$$

complex exponential form
of the signal $x(t)$

## Line Spectra

- The *amplitude spectrum* of $x(t)$ is defined as the plot of the magnitudes $|c_k|$ versus $\omega$
- The *phase spectrum* of $x(t)$ is defined as the plot of the angles $\angle c_k = \arg(c_k)$ versus $\omega$
- This results in *line spectra* which are defined for both positive and negative frequencies
- Notice: for $k = 1, 2, \ldots$

$$|c_k| = |c_{-k}| \qquad \angle c_k = -\angle c_{-k}$$

$$\arg(c_k) = -\arg(c_{-k})$$

## Example: Line Spectra

$$x(t) = \cos(t) + 0.5\cos(4t + \pi/3) + \cos(8t + \pi/2)$$

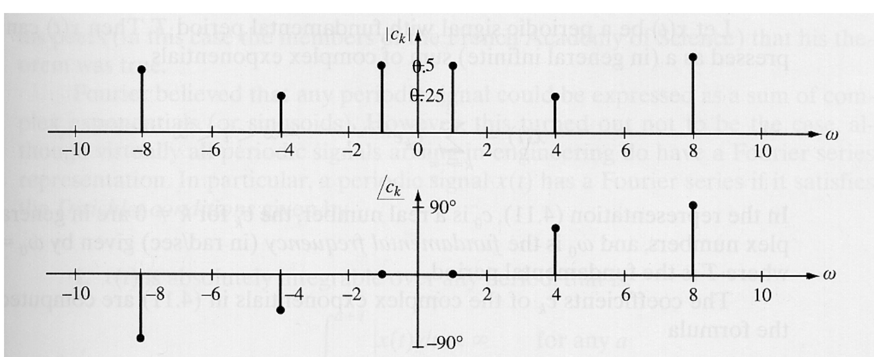



**Figure 4.5** Line spectra for the signal in Example 4.2.

## Fourier Series Representation of Periodic Signals

- Let $x(t)$ be a CT periodic signal with period $T$, *i.e.*, $x(t+T) = x(t), \quad \forall t \in R$

- Example: the rectangular pulse train

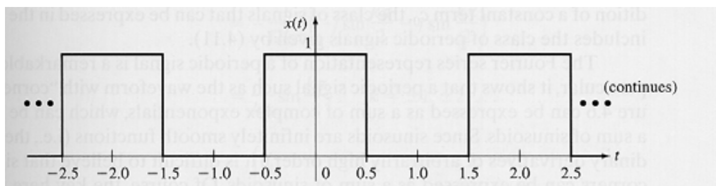



**Figure 4.6** Periodic signal with fundamental period $T = 2$.

## The Fourier Series

- Then, $x(t)$ can be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad t \in \mathbb{R}$$

where $\omega_0 = 2\pi / T$ is the *fundamental frequency* (*rad/sec*) of the signal and

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_o t} dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

$c_0$ is called the *constant or dc component* of $x(t)$

## The Fourier Series – Cont'd

- The frequencies $k\omega_0$ present in $x(t)$ are integer multiples of the fundamental frequency $\omega_0$
- Notice that, if the dc term $c_0$ is added to

$$x(t) = \sum_{\substack{k=-N \\ k \neq 0}}^{N} c_k e^{j\omega_k t}$$

and we set $N = \infty$, the Fourier series is a special case of the above equation where all the frequencies are integer multiples of $\omega_0$

## Dirichlet Conditions

- A periodic signal $x(t)$, has a Fourier series if it satisfies the following conditions:

1. $x(t)$ is absolutely integrable over any period, namely

$$\int_a^{a+T} |x(t)|\, dt < \infty, \quad \forall a \in \mathbb{R}$$

2. $x(t)$ has only a finite number of maxima and minima over any period
3. $x(t)$ has only a finite number of discontinuities over any period

## Example: The Rectangular Pulse Train

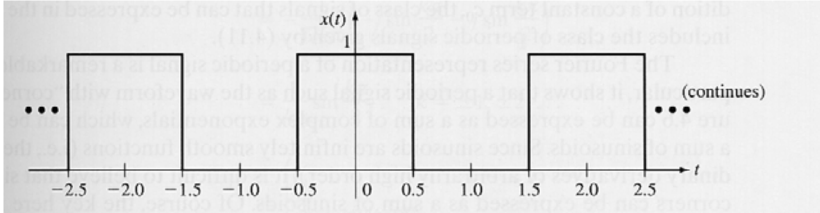



**Figure 4.6** Periodic signal with fundamental period $T = 2$.

- From figure, $T = 2$ whence $\omega_0 = 2\pi / 2 = \pi$
- Clearly $x(t)$ satisfies the Dirichlet conditions and thus has a Fourier series representation

## Example: The Rectangular Pulse Train – Cont'd

$$x(t) = \frac{1}{2} + \sum_{\substack{k=-\infty \\ k\ odd}}^{\infty} \frac{1}{k\pi}(-1)^{|(k-1)/2|} e^{jk\pi t}, \quad t \in \mathbb{R}$$

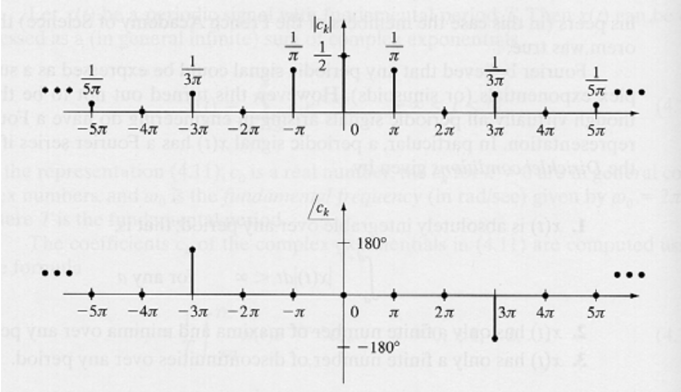



**Figure 4.7** Line spectra for the rectangular pulse train.

## Trigonometric Fourier Series

- By using Euler's formula, we can rewrite

$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad t\in\mathbb{R}$$

as

$$x(t)=\underbrace{c_0}_{\text{dc component}}+\sum_{k=1}^{\infty}\underbrace{2\,|c_k|\cos(k\omega_0 t+\angle c_k)}_{k\text{-th harmonic}}, \quad t\in\mathbb{R}$$

- This expression is called the trigonometric Fourier series of $x(t)$

## Example: Trigonometric Fourier Series of the Rectangular Pulse Train

- The expression

$$x(t)=\frac{1}{2}+\sum_{\substack{k=-\infty \\ k\ odd}}^{\infty}\frac{1}{k\pi}(-1)^{|(k-1)/2|}e^{jk\pi t}, \quad t\in\mathbb{R}$$

can be rewritten as

$$x(t)=\frac{1}{2}+\sum_{\substack{k=1 \\ k\ odd}}^{\infty}\frac{2}{k\pi}\cos\left(k\pi t+\left[(-1)^{(k-1)/2}-1\right]\frac{\pi}{2}\right), \quad t\in\mathbb{R}$$

## Gibbs Phenomenon

- Given an odd positive integer $N$, define the $N$-th partial sum of the previous series

$$x_N(t) = \frac{1}{2} + \sum_{\substack{k=1 \\ k\ odd}}^{N} \frac{2}{k\pi} \cos\left( k\pi t + \left[ (-1)^{(k-1)/2} - 1 \right] \frac{\pi}{2} \right), \quad t \in \mathbb{R}$$

- According to Fourier's theorem, it should be

$$\lim_{N \to \infty} | x_N(t) - x(t) | = 0$$

## Gibbs Phenomenon – Cont'd

$x_3(t)$

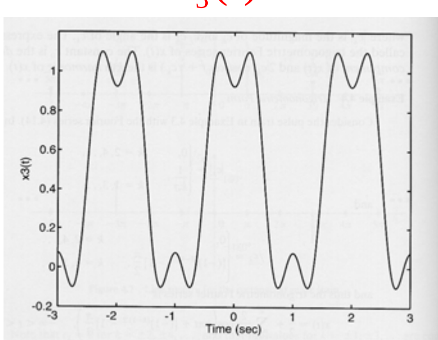



Figure 4.8 Plot of $x_N(t)$ when $N = 3$.

$x_9(t)$

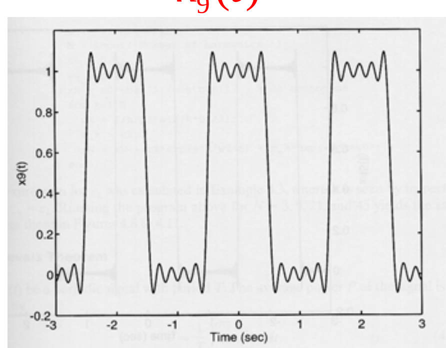



Figure 4.9 Approximation $x_9(t)$.

## Gibbs Phenomenon – Cont'd

$x_{21}(t)$

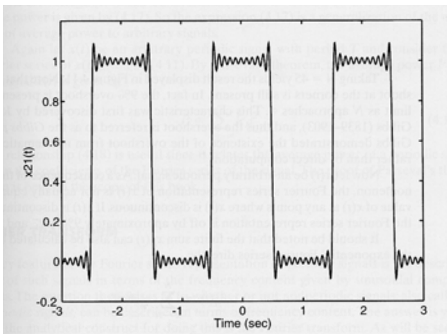



Figure 4.10 Approximation $x_{21}(t)$.

$x_{45}(t)$

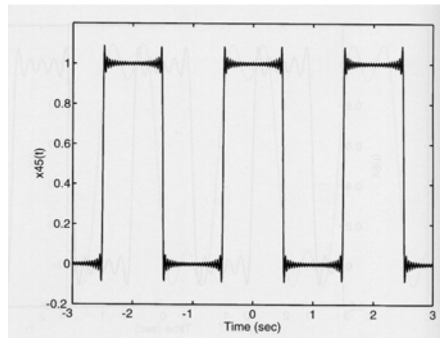



Figure 4.11 The signal $x_{45}(t)$.

overshoot: about 9 % of the signal magnitude (present even if $N \to \infty$)

## Parseval's Theorem

- Let $x(t)$ be a periodic signal with period $T$
- The *average power* $P$ of the signal is defined as

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt$$

- Expressing the signal as $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad t \in \mathbb{R}$ it is also

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2$$

## Fourier Transform

- We have seen that periodic signals can be represented with the Fourier series
- Can aperiodic signals be analyzed in terms of frequency components?
- Yes, and the Fourier transform provides the tool for this analysis
- The major difference w.r.t. the line spectra of periodic signals is that the spectra of aperiodic signals are defined for all real values of the frequency variable $\omega$ not just for a discrete set of values

## Frequency Content of the Rectangular Pulse

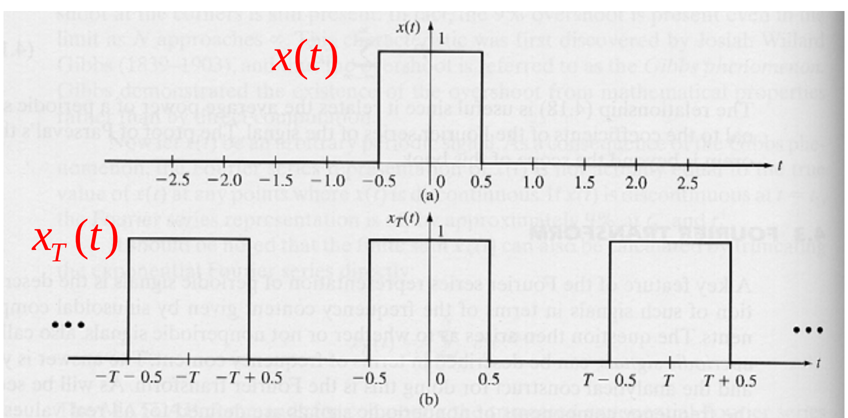



**Figure 4.12** Plots of the (a) one-second rectangular pulse and (b) pulse train.

$$x(t) = \lim_{T \to \infty} x_T(t)$$

## Frequency Content of the Rectangular Pulse – Cont'd

- Since $x_T(t)$ is periodic with period $T$, we can write

$$x_T(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad t \in \mathbb{R}$$

where

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_o t} dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

## Frequency Content of the Rectangular Pulse – Cont'd

- What happens to the frequency components of $x_T(t)$ as $T \to \infty$?
- For $k = 0$

$$c_0 = \frac{1}{T}$$

- For $k \neq 0$

$$c_k = \frac{2}{k\omega_0 T} \sin\left(\frac{k\omega_0}{2}\right) \underset{\omega_0 = 2\pi / T}{=} \frac{1}{k\pi} \sin\left(\frac{k\omega_0}{2}\right), \quad k = \pm 1, \pm 2, \ldots$$

## Frequency Content of the Rectangular Pulse – Cont'd

plots of $T\,|c_k|$
vs. $\omega = k\omega_0$
for $T = 2,5,10$

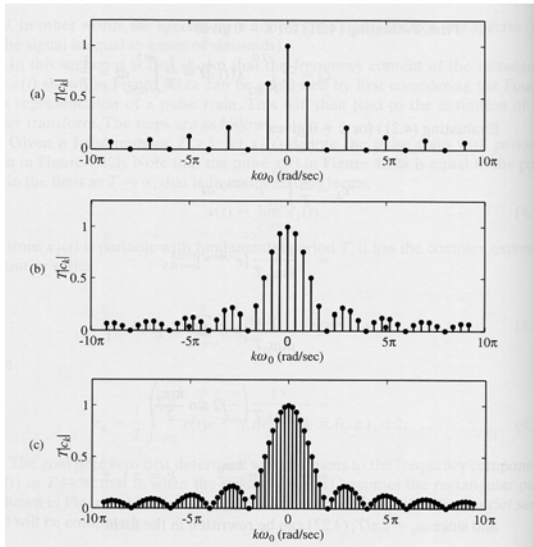



**Figure 4.13** Plot of scaled spectrum of $x_T(t)$ for (a) $T = 2$, (b) $T = 5$, and (c) $T = 10$.

## Frequency Content of the Rectangular Pulse – Cont'd

- It can be easily shown that

$$\lim_{T\to\infty} Tc_k = \operatorname{sinc}\left(\frac{\omega}{2\pi}\right), \quad \omega \in \mathbb{R}$$

where

$$\operatorname{sinc}(\lambda) \doteq \frac{\sin(\pi\lambda)}{\pi\lambda}$$

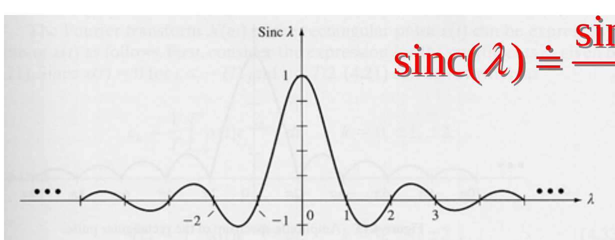



**Figure 4.14** Plot of sinc $\lambda$.

## Fourier Transform of the Rectangular Pulse

- The Fourier transform of the rectangular pulse $x(t)$ is defined to be the limit of $Tc_k$ as $T \to \infty$, i.e.,

$$X(\omega) = \lim_{T\to\infty} Tc_k = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right), \quad \omega \in \mathbb{R}$$

$|X(\omega)|$

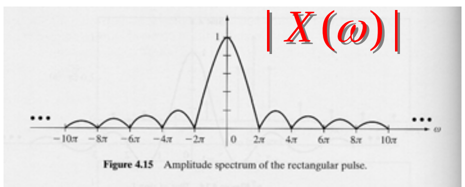



Figure 4.15 Amplitude spectrum of the rectangular pulse.

$\arg(X(\omega))$

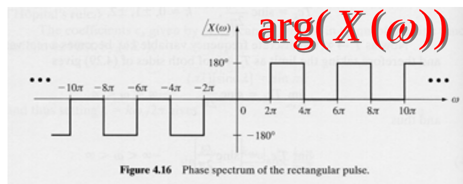



Figure 4.16 Phase spectrum of the rectangular pulse.

## Fourier Transform of the Rectangular Pulse – Cont'd

- The Fourier transform $X(\omega)$ of the rectangular pulse $x(t)$ can be expressed in terms of $x(t)$ as follows:

$$c_k = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_o t}dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

$x(t) = 0$ for $t < -T/2$ and $t > T/2$

whence

$$Tc_k = \int_{-\infty}^{\infty} x(t)e^{-jk\omega_o t}dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

## Fourier Transform of the Rectangular Pulse – Cont'd

- Now, by definition $X(\omega) = \lim_{T \to \infty} Tc_k$ and, since $k\omega_0 \to \omega$ as $T \to \infty$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \quad \omega \in \mathbb{R}$$

- The *inverse Fourier transform* of $X(\omega)$ is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega, \quad t \in \mathbb{R}$$

## The Fourier Transform in the General Case

- Given a signal $x(t)$, its *Fourier transform* $X(\omega)$ is defined as

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \quad \omega \in \mathbb{R}$$

- A signal $x(t)$ is said to have a *Fourier transform in the ordinary sense* if the above integral converges

## The Fourier Transform in the General Case – Cont'd

- The integral does converge if
  1. the signal $x(t)$ is "*well-behaved*"
  2. and $x(t)$ is *absolutely integrable*, namely,

$$\int_{-\infty}^{\infty} | x(t) | \, dt < \infty$$

- Note: *well behaved* means that the signal has a finite number of discontinuities, maxima, and minima within any finite time interval

## Example: The DC or Constant Signal

- Consider the signal $x(t) = 1, \quad t \in \mathbb{R}$
- Clearly $x(t)$ does not satisfy the first requirement since

$$\int_{-\infty}^{\infty} | x(t) | \, dt = \int_{-\infty}^{\infty} dt = \infty$$

- Therefore, the constant signal does not have a *Fourier transform in the ordinary sense*
- Later on, we'll see that it has however a *Fourier transform in a generalized sense*

## Example: The Exponential Signal

- Consider the signal $x(T) = e^{-bt}u(t), \quad b \in \mathbb{R}$
- Its Fourier transform is given by

$$X(\omega) = \int_{-\infty}^{\infty} e^{-bt}u(t)e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-(b+j\omega)t}dt = -\frac{1}{b+j\omega}\left[e^{-(b+j\omega)t}\right]_{t=0}^{t=\infty}$$

## Example: The Exponential Signal – Cont'd

- If $b < 0$ , $X(\omega)$ does not exist
- If $b = 0$ , $x(t) = u(t)$ and $X(\omega)$ does not exist either in the ordinary sense
- If $b > 0$ , it is

$$X(\omega) = \frac{1}{b+j\omega}$$

amplitude spectrum

$$|X(\omega)| = \frac{1}{\sqrt{b^2+\omega^2}}$$

phase spectrum

$$\arg(X(\omega)) = -\arctan\left(\frac{\omega}{b}\right)$$

## Example: Amplitude and Phase Spectra of the Exponential Signal

$$x(t) = e^{-10t} u(t)$$

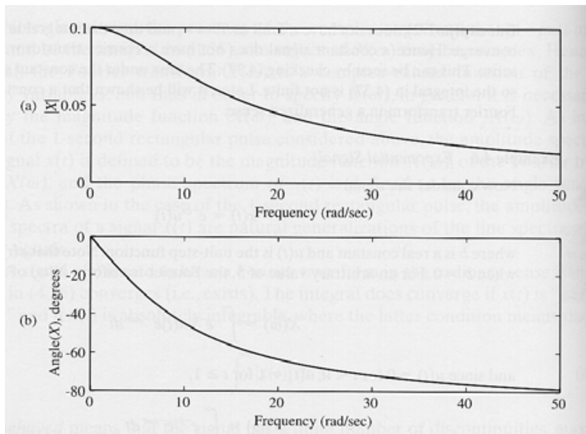



**Figure 4.17** Plots of the (a) amplitude and (b) phase spectra of $x(t) = \exp(-10t)u(t)$.

## Rectangular Form of the Fourier Transform

- Consider

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt, \quad \omega \in \mathbb{R}$$

- Since $X(\omega)$ in general is a complex function, by using Euler's formula

$$X(\omega) = \underbrace{\int_{-\infty}^{\infty} x(t)\cos(\omega t) dt}_{R(\omega)} + j \underbrace{\left( -\int_{-\infty}^{\infty} x(t)\sin(\omega t) dt \right)}_{I(\omega)}$$

$$X(\omega) = R(\omega) + jI(\omega)$$

## Polar Form of the Fourier Transform

- $X(\omega) = R(\omega) + jI(\omega)$ can be expressed in a polar form as

$$X(\omega) = |X(\omega)| \exp(j \arg(X(\omega)))$$

where

$$|X(\omega)| = \sqrt{R^2(\omega) + I^2(\omega)}$$

$$\arg(X(\omega)) = \arctan\left(\frac{I(\omega)}{R(\omega)}\right)$$

## Fourier Transform of Real-Valued Signals

- If $x(t)$ is real-valued, it is

$$X(-\omega) = X^*(\omega)$$

Hermitian symmetry

- Moreover

$$X^*(\omega) = |X(\omega)| \exp(-j \arg(X(\omega)))$$

whence

$$|X(-\omega)| = |X(\omega)| \quad \text{and}$$

$$\arg(X(-\omega)) = -\arg(X(\omega))$$

## Fourier Transforms of Signals with Even or Odd Symmetry

- Even signal: $x(t) = x(-t)$

$$X(\omega) = 2\int_0^{\infty} x(t)\cos(\omega t)\,dt$$

- Odd signal: $x(t) = -x(-t)$

$$X(\omega) = -j2\int_0^{\infty} x(t)\sin(\omega t)\,dt$$

## Example: Fourier Transform of the Rectangular Pulse

- Consider the even signal

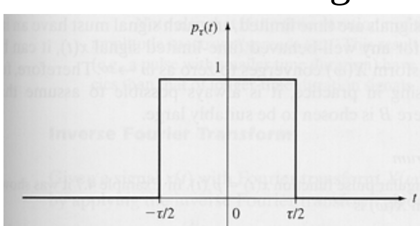



**Figure 4.18** Rectangular pulse of duration $\tau$ seconds.

- It is

$$X(\omega) = 2\int_0^{\tau/2} (1)\cos(\omega t)\,dt = \frac{2}{\omega}\left[\sin(\omega t)\right]_{t=0}^{t=\tau/2} = \frac{2}{\omega}\sin\left(\frac{\omega\tau}{2}\right)$$

$$= \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

## Example: Fourier Transform of the Rectangular Pulse – Cont'd

$$X(\omega) = \tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

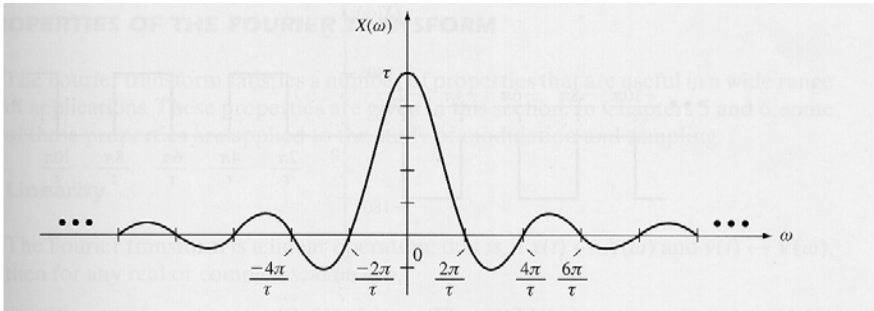



**Figure 4.19** Fourier transform of the $\tau$-second rectangular pulse.

## Example: Fourier Transform of the Rectangular Pulse – Cont'd

amplitude spectrum

phase spectrum

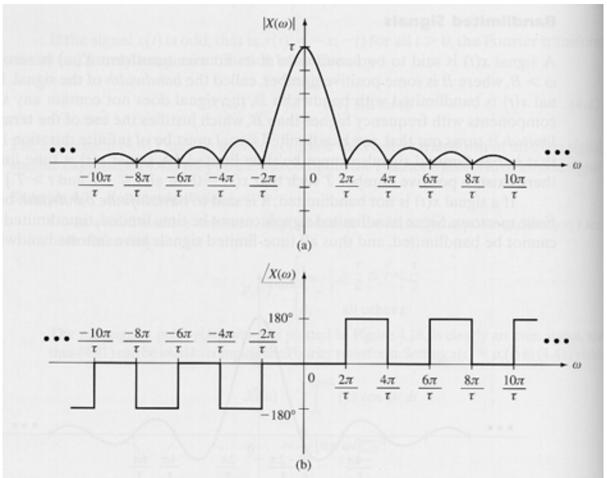



**Figure 4.20** (a) Amplitude and (b) phase spectra of the rectangular pulse.

## Bandlimited Signals

- A signal $x(t)$ is said to be *bandlimited* if its Fourier transform $X(\omega)$ is zero for all $\omega > B$ where $B$ is some positive number, called the *bandwidth of the signal*
- It turns out that any bandlimited signal must have an infinite duration in time, i.e., bandlimited signals cannot be time limited

## Bandlimited Signals – Cont'd

- If a signal $x(t)$ is not bandlimited, it is said to have *infinite bandwidth* or an *infinite spectrum*
- Time-limited signals cannot be bandlimited and thus all time-limited signals have infinite bandwidth
- However, for any well-behaved signal $x(t)$ it can be proven that $\lim_{\omega \to \infty} X(\omega) = 0$ whence it can be assumed that

$$|X(\omega)| \approx 0 \quad \forall \omega > B$$

*B* being a convenient large number

## Inverse Fourier Transform

- Given a signal $x(t)$ with Fourier transform $X(\omega)$, $x(t)$ can be recomputed from $X(\omega)$ by applying the inverse Fourier transform given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega, \quad t \in \mathbb{R}$$

- Transform pair

$$x(t) \leftrightarrow X(\omega)$$

## Properties of the Fourier Transform

$$x(t) \leftrightarrow X(\omega) \qquad y(t) \leftrightarrow Y(\omega)$$

- *Linearity:*

$$\alpha x(t) + \beta y(t) \leftrightarrow \alpha X(\omega) + \beta Y(\omega)$$

- *Left or Right Shift in Time:*

$$x(t - t_0) \leftrightarrow X(\omega)e^{-j\omega t_0}$$

- *Time Scaling:*

$$x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right)$$

## Properties of the Fourier Transform

- *Time Reversal:*

$$x(-t) \leftrightarrow X(-\omega)$$

- *Multiplication by a Power of t:*

$$t^n x(t) \leftrightarrow (j)^n \frac{d^n}{d\omega^n} X(\omega)$$

- *Multiplication by a Complex Exponential:*

$$x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

## Properties of the Fourier Transform

- *Multiplication by a Sinusoid (Modulation):*

$$x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}\left[X(\omega+\omega_0) - X(\omega-\omega_0)\right]$$

$$x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\left[X(\omega+\omega_0) + X(\omega-\omega_0)\right]$$

- *Differentiation in the Time Domain:*

$$\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega)$$

## Properties of the Fourier Transform

- *Integration in the Time Domain:*

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$$

- *Convolution in the Time Domain:*

$$x(t)*y(t) \leftrightarrow X(\omega)Y(\omega)$$

- *Multiplication in the Time Domain:*

$$x(t)y(t) \leftrightarrow X(\omega)*Y(\omega)$$

## Properties of the Fourier Transform

- *Parseval's Theorem:*

$$\int_{\mathbb{R}} x(t)y(t)dt \leftrightarrow \frac{1}{2\pi}\int_{\mathbb{R}} X^*(\omega)Y(\omega)d\omega$$

if $y(t)=x(t)$ $\quad \int_{\mathbb{R}} |x(t)|^2\,dt \leftrightarrow \frac{1}{2\pi}\int_{\mathbb{R}} |X(\omega)|^2 d\omega$

- *Duality:*

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

# Properties of the Fourier Transform - Summary

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

| Property | Transform Pair/Property |
| --- | --- |
| Linearity | $ax(t) + bv(t) \leftrightarrow aX(\omega) + bV(\omega)$ |
| Right or left shift in time | $x(t - c) \leftrightarrow X(\omega)e^{-j\omega c}$ |
| Time scaling | $x(at) \leftrightarrow \frac{1}{a}X\left(\frac{\omega}{a}\right) \quad a > 0$ |
| Time reversal | $x(-t) \leftrightarrow X(-\omega) = \overline{X(\omega)}$ |
| Multiplication by a power of t | $t^n x(t) \leftrightarrow j^n \frac{d^n}{d\omega^n} X(\omega) \quad n = 1, 2, \ldots$ |
| Multiplication by a complex exponential | $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0) \quad \omega_0$ real |
| Multiplication by $\sin \omega_0 t$ | $x(t) \sin \omega_0 t \leftrightarrow \frac{j}{2}[X(\omega + \omega_0) - X(\omega - \omega_0)]$ |
| Multiplication by $\cos \omega_0 t$ | $x(t) \cos \omega_0 t \leftrightarrow \frac{1}{2}[X(\omega + \omega_0) + X(\omega - \omega_0)]$ |
| Differentiation in the time domain | $\frac{d^n}{dt^n} x(t) \leftrightarrow (j\omega)^n X(\omega) \quad n = 1, 2, \ldots$ |
| Integration | $\int_{-\infty}^{t} x(\lambda)\, d\lambda \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$ |
| Convolution in the time domain | $x(t) * v(t) \leftrightarrow X(\omega)V(\omega)$ |
| Multiplication in the time domain | $x(t)v(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * V(\omega)$ |
| Parseval's theorem | $\int_{-\infty}^{\infty} x(t)v(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)\, d\omega$ |
| Special case of Parseval's theorem | $\int_{-\infty}^{\infty} x^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\, d\omega$ |
| Duality | $X(t) \leftrightarrow 2\pi x(-\omega)$ |

# Example: Linearity

$$x(t) = p_4(t) + p_2(t)$$

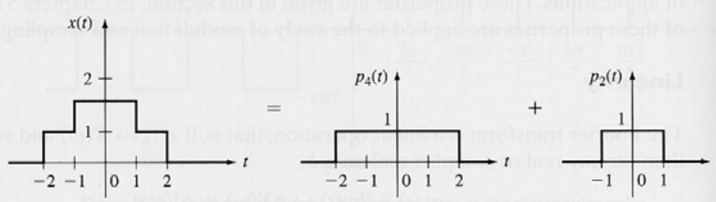



**Figure 4.21** Signal in Example 4.9.

$$X(\omega) = 4\operatorname{sinc}\left(\frac{2\omega}{\pi}\right) + 2\operatorname{sinc}\left(\frac{\omega}{\pi}\right)$$

## Example: Time Shift

$$x(t) = p_2(t-1)$$

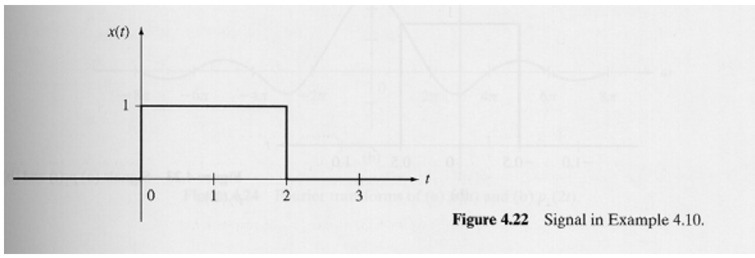



Figure 4.22 Signal in Example 4.10.

$$X(\omega) = 2\mathrm{sinc}\left(\frac{\omega}{\pi}\right)e^{-j\omega}$$

## Example: Time Scaling

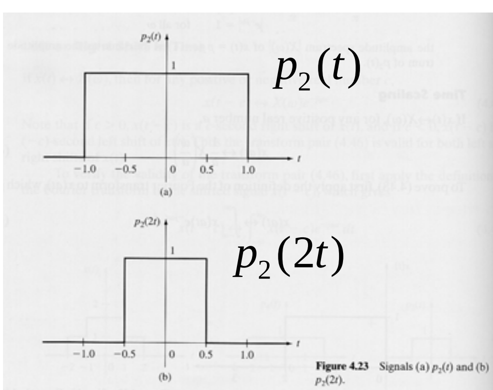



Figure 4.23 Signals (a) $p_2(t)$ and (b) $p_2(2t)$.

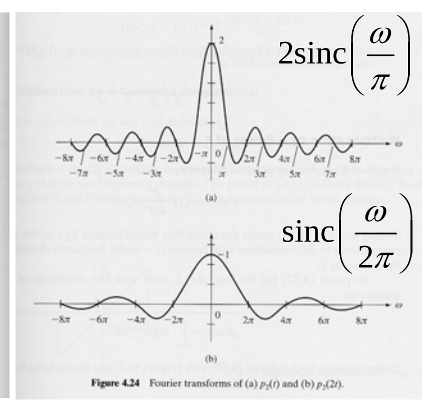



Figure 4.24 Fourier transforms of (a) $p_2(t)$ and (b) $p_2(2t)$.

$a > 1$ time compression $\leftrightarrow$ frequency expansion

$0 < a < 1$ time expansion $\leftrightarrow$ frequency compression

## Example: Multiplication in Time

$$x(t) = tp_2(t)$$

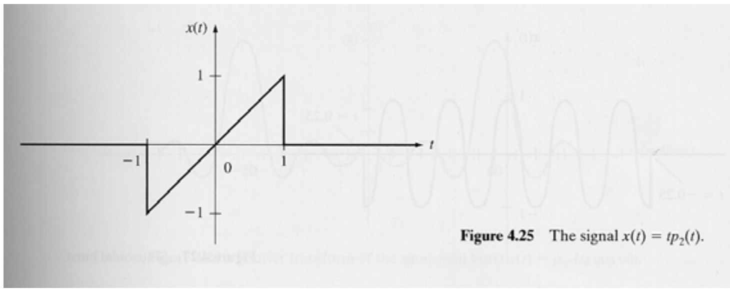



Figure 4.25 The signal $x(t) = tp_2(t)$.

$$X(\omega) = j\frac{d}{d\omega}\left(2\operatorname{sinc}\left(\frac{\omega}{\pi}\right)\right) = j2\frac{d}{d\omega}\left(\frac{\sin\omega}{\omega}\right) = j2\frac{\omega\cos\omega - \sin\omega}{\omega^2}$$

## Example: Multiplication in Time – Cont'd

$$X(\omega) = j2\frac{\omega\cos\omega - \sin\omega}{\omega^2}$$

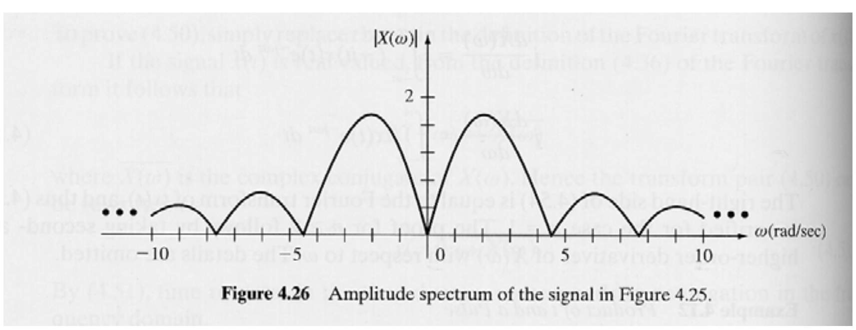



Figure 4.26 Amplitude spectrum of the signal in Figure 4.25.

## Example: Multiplication by a Sinusoid

$$x(t) = p_\tau(t)\cos(\omega_0 t)$$ sinusoidal burst

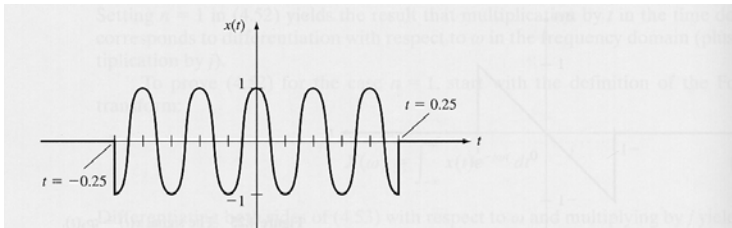



Figure 4.27 Sinusoidal burst.

$$X(\omega) = \frac{1}{2}\left[\tau\,\mathrm{sinc}\left(\frac{\tau(\omega+\omega_0)}{2\pi}\right) + \tau\,\mathrm{sinc}\left(\frac{\tau(\omega-\omega_0)}{2\pi}\right)\right]$$

## Example: Multiplication by a Sinusoid – Cont'd

$$X(\omega) = \frac{1}{2}\left[\tau\,\mathrm{sinc}\left(\frac{\tau(\omega+\omega_0)}{2\pi}\right) + \tau\,\mathrm{sinc}\left(\frac{\tau(\omega-\omega_0)}{2\pi}\right)\right]$$

$$\begin{cases} \omega_0 = 60\ rad/\sec \\ \tau = 0.5 \end{cases}$$

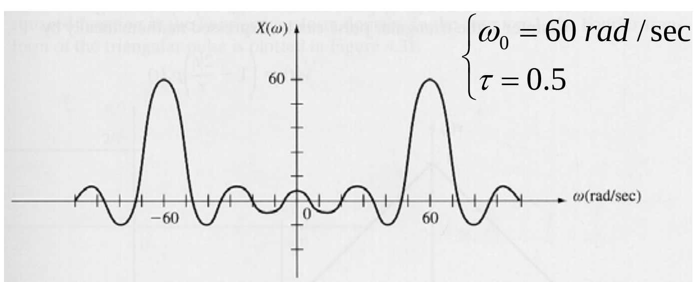
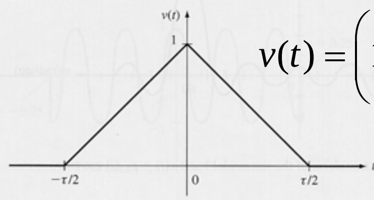


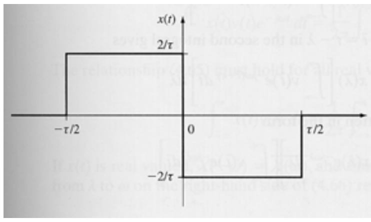

Figure 4.28 Fourier transform of the sinusoidal burst $x(t) = p_{0.5}(t)\cos 60t$.

## Example: Integration in the Time Domain




$$v(t) = \left(1 - \frac{2\,|\,t\,|}{\tau}\right) p_{\tau}(t)$$

**Figure 4.29** Triangular pulse.




$$x(t) = \frac{dv(t)}{dt}$$

**Figure 4.30** Derivative of the triangular pulse.

## Example: Integration in the Time Domain – Cont'd

- The Fourier transform of $x(t)$ can be easily found to be

$$X(\omega) = \left(\operatorname{sinc}\left(\frac{\tau\omega}{4\pi}\right)\right)\left(j2\sin\left(\frac{\tau\omega}{4}\right)\right)$$

- Now, by using the integration property, it is

$$V(\omega) = \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega) = \frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$$

## Example: Integration in the Time Domain – Cont'd

$$V(\omega) = \frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\tau\omega}{4\pi}\right)$$

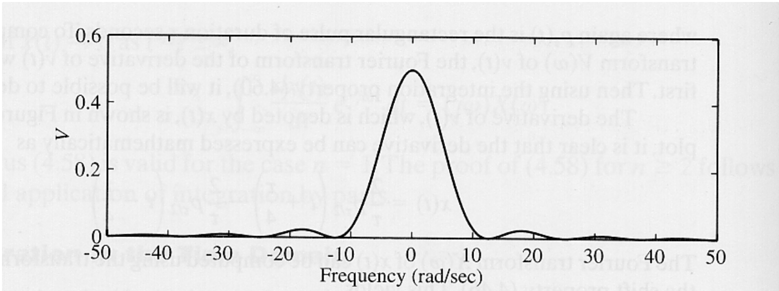



**Figure 4.31** Fourier transform of the 1-second triangular pulse.

## Generalized Fourier Transform

- Fourier transform of $\delta(t)$

$$\int_{\mathbb{R}} \delta(t)e^{-j\omega t}dt = 1 \quad \Longrightarrow \quad \delta(t) \leftrightarrow 1$$

- Applying the duality property

$$x(t) = 1,\, t \in \mathbb{R} \leftrightarrow \underbrace{2\pi\delta(\omega)}$$

*generalized Fourier transform*
of the constant signal $x(t) = 1,\, t \in \mathbb{R}$

## Generalized Fourier Transform of Sinusoidal Signals

$$\cos(\omega_0 t) \leftrightarrow \pi\left[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)\right]$$

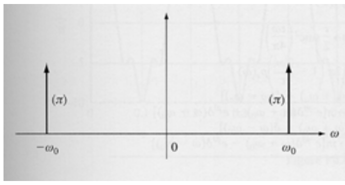



**Figure 4.32** Fourier transform of $\cos \omega_0 t$.

$$\sin(\omega_0 t) \leftrightarrow j\pi\left[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)\right]$$

## Fourier Transform of Periodic Signals

- Let $x(t)$ be a periodic signal with period $T$; as such, it can be represented with its Fourier transform

$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \qquad \omega_0 = 2\pi / T$$

- Since $e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$, it is

$$X(\omega)=\sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega-k\omega_0)$$

## Fourier Transform of the Unit-Step Function

- Since

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

using the integration property, it is

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \leftrightarrow \frac{1}{j\omega} + \pi\delta(\omega)$$

## Common Fourier Transform Pairs

**TABLE 4.2** COMMON FOURIER TRANSFORM PAIRS

$1, \quad -\infty < t < \infty \leftrightarrow 2\pi\delta(\omega)$

$-0.5 + u(t) \leftrightarrow \dfrac{1}{j\omega}$

$u(t) \leftrightarrow \pi\delta(\omega) + \dfrac{1}{j\omega}$

$\delta(t) \leftrightarrow 1$

$\delta(t - c) \leftrightarrow e^{-j\omega c}, \quad c$ any real number

$e^{-bt}u(t) \leftrightarrow \dfrac{1}{j\omega + b}, \quad b > 0$

$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$, $\omega_0$ any real number

$p_\tau(t) \leftrightarrow \tau \operatorname{sinc} \dfrac{\tau\omega}{2\pi}$

$\tau \operatorname{sinc} \dfrac{\tau t}{2\pi} \leftrightarrow 2\pi p_\tau(\omega)$

$\left(1 - \dfrac{2|t|}{\tau}\right)p_\tau(t) \leftrightarrow \dfrac{\tau}{2} \operatorname{sinc}^2\left(\dfrac{\tau\omega}{4\pi}\right)$

$\dfrac{\tau}{2} \operatorname{sinc}^2\left(\dfrac{\tau t}{4\pi}\right) \leftrightarrow 2\pi\left(1 - \dfrac{2|\omega|}{\tau}\right)p_\tau(\omega)$

$\cos \omega_0 t \leftrightarrow \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$

$\cos(\omega_0 t + \theta) \leftrightarrow \pi[e^{-j\theta}\delta(\omega + \omega_0) + e^{j\theta}\delta(\omega - \omega_0)]$

$\sin \omega_0 t \leftrightarrow j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$

$\sin(\omega_0 t + \theta) \leftrightarrow j\pi[e^{-j\theta}\delta(\omega + \omega_0) - e^{j\theta}\delta(\omega - \omega_0)]$