# Chapter 2
# Systems Defined by Differential or Difference Equations

# Linear I/O Differential Equations with Constant Coefficients

- Consider the CT SISO system

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

described by

$$y^{(N)} + \sum_{i=0}^{N-1} a_i y^{(i)}(t) = \sum_{i=0}^{M} b_i x^{(i)}(t)$$

$$a_i \in \mathbb{R}, b_i \in \mathbb{R} \qquad y^{(i)}(t) \doteq \frac{d^i y(t)}{dt^i} \qquad x^{(i)}(t) \doteq \frac{d^i x(t)}{dt^i}$$

## Initial Conditions

- In order to solve the previous equation for $t > 0$, we have to know the $N$ initial conditions

$$y(0), y^{(1)}(0), \ldots, y^{(N-1)}(0)$$

## Initial Conditions – Cont'd

- If the $M$-th derivative of the input $x(t)$ contains an impulse $k\delta(t)$ or a derivative of an impulse, the N initial conditions must be taken at time $t = 0^-$, i.e.,

$$y(0^-), y^{(1)}(0^-), \ldots, y^{(N-1)}(0^-)$$

## First-Order Case

- Consider the following differential equation:

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

- Its solution is

$$y(t) = y(0)e^{-at} + \int_0^t e^{-a(t-\tau)} bx(\tau)\,d\tau, \quad t \geq 0$$

or

$$y(t) = y(0^-)e^{-at} + \int_{0^-}^t e^{-a(t-\tau)} bx(\tau)\,d\tau, \quad t \geq 0$$

if the initial time is taken to be $0^-$

## Generalization of the First-Order Case

- Consider the equation:

$$\frac{dy(t)}{dt} + ay(t) = b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

- Define $q(t) = y(t) - b_1 x(t)$
- Differentiating this equation, we obtain

$$\frac{dq(t)}{dt} = \frac{dy(t)}{dt} - b_1 \frac{dx(t)}{dt}$$

## Generalization of the First-Order Case – Cont'd

$$\frac{\cancel{dy(t)}}{dt} + ay(t) = b_1 \frac{\cancel{dx(t)}}{dt} + b_0 x(t)$$

$$+$$

$$\frac{dq(t)}{dt} = \frac{\cancel{dy(t)}}{dt} - b_1 \frac{\cancel{dx(t)}}{dt}$$

$$=$$

$$\frac{dq(t)}{dt} = -ay(t) + b_0 x(t)$$

## Generalization of the First-Order Case – Cont'd

- Solving $q(t) = y(t) - b_1 x(t)$ for $y(t)$ it is

$$y(t) = q(t) + b_1 x(t)$$

which, plugged into $\frac{dq(t)}{dt} = -ay(t) + b_0 x(t)$, yields

$$\frac{dq(t)}{dt} = -a\left(q(t) + b_1 x(t)\right) + b_0 x(t) =$$

$$= -aq(t) + (b_0 - ab_1)x(t)$$

## Generalization of the First-Order Case – Cont'd

If the solution of $\dfrac{dy(t)}{dt} + ay(t) = bx(t)$

is $y(t) = y(0)e^{-at} + \int_0^t e^{-a(t-\tau)} bx(\tau)d\tau, \quad t \ge 0$

then the solution of $\dfrac{dq(t)}{dt} = -aq(t) + (b_0 - ab_1)x(t)$

is

$$q(t) = q(0)e^{-at} + \int_0^t e^{-a(t-\tau)} (b_0 - ab_1)x(\tau)d\tau, \quad t \ge 0$$

## System Modeling – Electrical Circuits

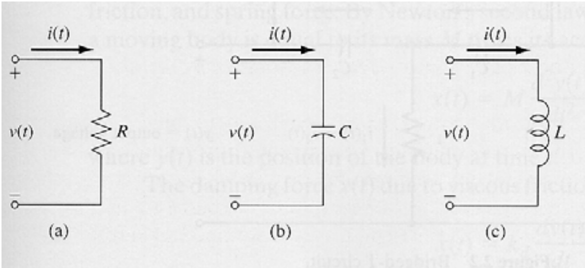

Figure 2.1 Basic circuit elements: (a) resistor; (b) capacitor; (c) inductor.

resistor $v(t) = Ri(t)$

capacitor $\dfrac{dv(t)}{dt} = \dfrac{1}{C} i(t) \quad or \quad v(t) = \dfrac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$

inductor $v(t) = L\dfrac{di(t)}{dt} \quad or \quad i(t) = \dfrac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$

# Example: Bridged-T Circuit

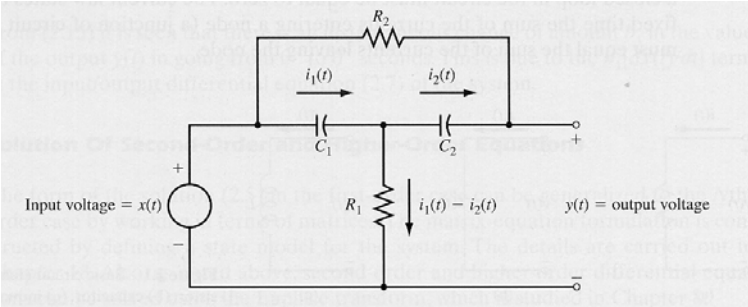


**Figure 2.2** Bridged-T circuit.

Kirchhoff's voltage law

loop (or mesh) equations

$$\begin{cases} v_1(t) + R_1\big(i_1(t) - i_2(t)\big) = x(t) \\ v_1(t) + v_2(t) + R_2 i_2(t) = 0 \\ y(t) = x(t) + R_2 i_2(t) \end{cases}$$

## Mechanical Systems

- Newton's second Law of Motion:

$$x(t) = M \frac{d^2 y(t)}{dt^2}$$

- Viscous friction:

$$x(t) = k_d \frac{dy(t)}{dt}$$

- Elastic force:

$$x(t) = k_s y(t)$$

# Example: Automobile Suspension System

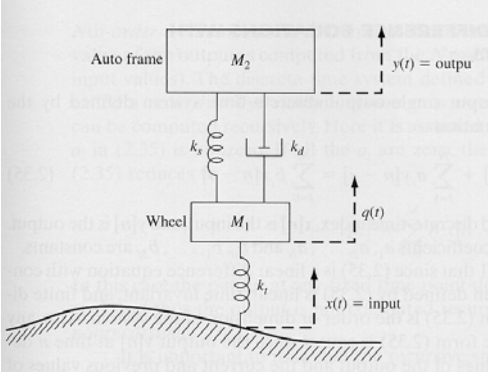


**Figure 2.5** Automobile suspension system.

$$\begin{cases} M_1 \dfrac{d^2q(t)}{dt^2} + k_t[q(t) - x(t)] = k_s[y(t) - q(t)] + k_d\left[\dfrac{dy(t)}{dt} - \dfrac{dq(t)}{dt}\right] \\ M_2 \dfrac{d^2y(t)}{dt^2} + k_s[y(t) - q(t)] + k_d\left[\dfrac{dy(t)}{dt} - \dfrac{dq(t)}{dt}\right] = 0 \end{cases}$$

## Rotational Mechanical Systems

- Inertia torque:

$$x(t) = I\frac{d^2\theta(t)}{dt^2}$$

- Damping torque:

$$x(t) = k_d\frac{d\theta(t)}{dt}$$

- Spring torque:

$$x(t) = k_s\theta(t)$$

## Linear I/O Difference Equation With Constant Coefficients

- Consider the DT SISO system

$$x[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n]$$

described by

$$y[n] + \sum_{i=1}^{N} a_i y[n-i] = \sum_{i=0}^{M} b_i x[n-i]$$

$a_i \in \mathbb{R}, b_i \in \mathbb{R}$ $\quad$ $N$ is the order or dimension of the system

## Solution by Recursion

- Unlike linear I/O differential equations, linear I/O difference equations can be solved by direct numerical procedure (*N*-th order recursion)

$$y[n] = -\sum_{i=1}^{N} a_i y[n-i] + \sum_{i=0}^{M} b_i x[n-i]$$

(*recursive DT system* or *recursive digital filter*)

## Solution by Recursion – Cont'd

- The solution by recursion for $n \geq 0$ requires the knowledge of the $N$ initial conditions

$$y[-N], y[-N+1], \ldots, y[-1]$$

and of the $M$ initial input values

$$x[-M], x[-M+1], \ldots, x[-1]$$

## Analytical Solution

- Like the solution of a constant-coefficient differential equation, the solution of

$$y[n] = -\sum_{i=1}^{N} a_i y[n-i] + \sum_{i=0}^{M} b_i x[n-i]$$

can be obtained analytically in a closed form and expressed as

$$y[n] = y_{zi}[n] + y_{zs}[n]$$

(total response = zero-input response + zero-state response)

- Solution method presented in ECE 464/564