Chapter 1
Fundamental Concepts

Signals

- A signal is a pattern of variation of a physical quantity as a *function* of time, space, distance, position, temperature, pressure, etc.
- These quantities are usually the independent variables of the function defining the signal
- A signal encodes information, which is the variation itself

Signal Processing

- Signal processing is the discipline concerned with extracting, analyzing, and manipulating the information carried by signals
- The processing method depends on the type of signal and on the nature of the information carried by the signal

Characterization and Classification of Signals

- The type of signal depends on the nature of the independent variables and on the value of the function defining the signal
- For example, the independent variables can be continuous or discrete
- Likewise, the signal can be a continuous or discrete function of the independent variables

Characterization and Classification of Signals – Cont'd

- Moreover, the signal can be either a real-valued function or a complex-valued function
- A signal consisting of a single component is called a scalar or one-dimensional (1-D) signal
- A signal consisting of multiple components is called a vector or multidimensional (M-D) signal

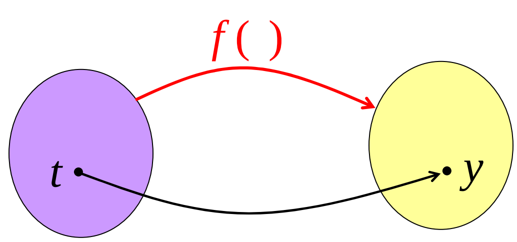
Definition of Function from Calculus

$$y = f(t)$$

$$f : t \mapsto y = f(t)$$

independent variable

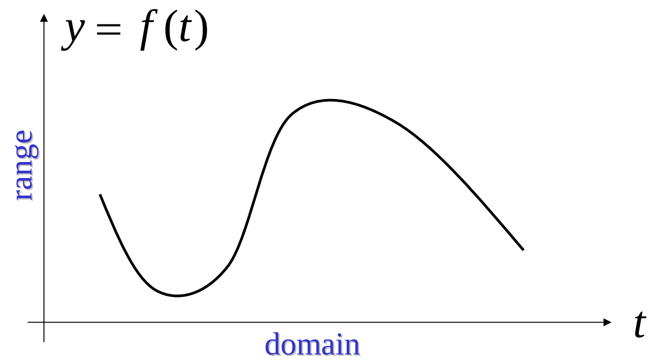
dependent variable



domain: set of values that t can take on

range: set of values spanned by y

Plot or Graph of a Function



Continuous-Time (CT) and Discrete-Time (DT) Signals

- A signal $x(t)$ depending on a continuous temporal variable $t \in \mathbb{R}$ will be called a continuous-time (CT) signal

- A signal $x[n]$ depending on a discrete temporal variable $n \in \mathbb{Z}$ will be called a discrete-time (DT) signal

Examples: CT vs. DT Signals

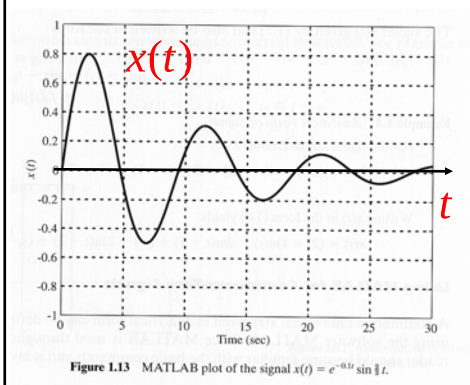


Figure 1.13 MATLAB plot of the signal $x(t) = e^{-0.1t}\sin\frac{2}{3}t$.

plot(t,x)

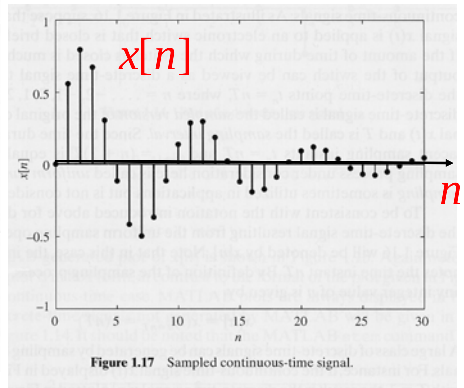


Figure 1.17 Sampled continuous-time signal.

stem(n,x)

CT Signals: 1-D vs. N-D, Real vs. Complex

- ✓ 1-D, real-valued, CT signal: $x(t) \in \mathbb{R}, t \in \mathbb{R}$
- N-D, real-valued, CT signal: $x(t) \in \mathbb{R}^N, t \in \mathbb{R}$
- ✓ 1-D, complex-valued, CT signal: $x(t) \in \mathbb{C}, t \in \mathbb{R}$
- N-D, complex-valued, CT signal: $x(t) \in \mathbb{C}^N, t \in \mathbb{R}$

DT Signals:
1-D vs. N-D, Real vs. Complex

- ✓ 1-D, real-valued, DT signal: $x[n] \in \mathbb{R}, n \in \mathbb{Z}$
- N-D, real-valued, DT signal: $x[n] \in \mathbb{R}^N, n \in \mathbb{Z}$
- ✓ 1-D, complex-valued, DT signal: $x[n] \in \mathbb{C}, n \in \mathbb{Z}$
- N-D, complex-valued, DT signal: $x[n] \in \mathbb{C}^N, n \in \mathbb{Z}$

Digital Signals

- A DT signal whose values belong to a finite set or alphabet $\mathrm{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ is called a digital signal
- Since computers work with finite-precision arithmetic, only digital signals can be numerically processed
- Digital Signal Processing (DSP): ECE 464/564 (Liu) and ECE 567 (Lucchese)

Digital Signals: 1-D vs. N-D

- 1-D, real-valued, digital signal: $x[n] \in \mathrm{A},\ n \in \mathbb{Z}$
- N-D, real-valued, digital signal: $x[n] \in \mathrm{A}^N,\ n \in \mathbb{Z}$

$$\mathrm{A} = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$$

If $\alpha_i \in \mathbb{R}$, the digital signal is real, if instead at least one of the $\alpha_i \in \mathbb{C}$, the digital signal is complex

Systems

- A system is any device that can process signals for analysis, synthesis, enhancement, format conversion, recording, transmission, etc.
- A system is usually mathematically defined by the equation(s) relating input to output signals (I/O characterization)
- A system may have single or multiple inputs and single or multiple outputs

Block Diagram Representation of Single-Input Single-Output (SISO) CT Systems

input signal

output signal

$$x(t) \longrightarrow \boxed{T} \longrightarrow y(t) = T\{x(t)\}$$

$t \in \mathbb{R}$ $\qquad$ $t \in \mathbb{R}$

Block Diagram Representation of Single-Input Single-Output (SISO) DT Systems

input signal

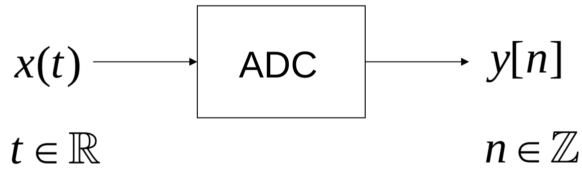
output signal

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n] = T\{x[n]\}$$

$n \in \mathbb{Z}$ $\qquad$ $n \in \mathbb{Z}$

A Hybrid SISO System: The Analog to Digital Converter (ADC)

Used to convert a CT (analog) signal into a digital signal



Block Diagram Representation of Multiple-Input Multiple-Output (MIMO) CT Systems

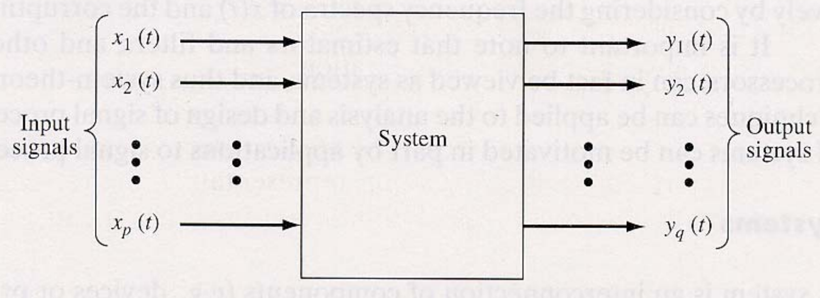
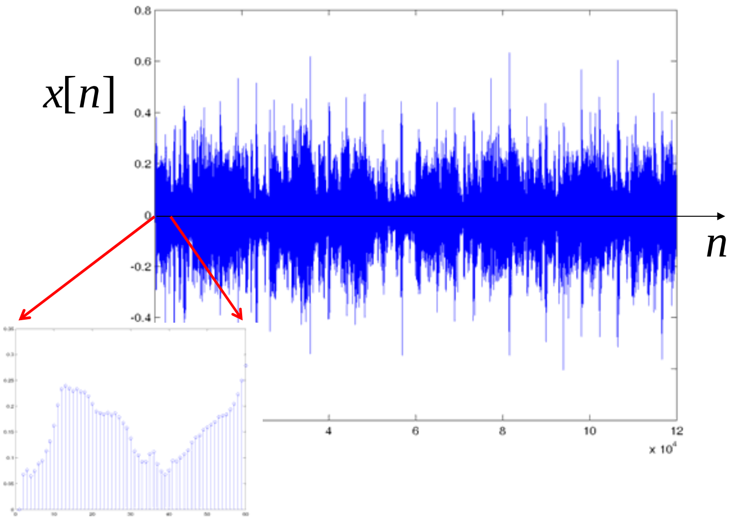
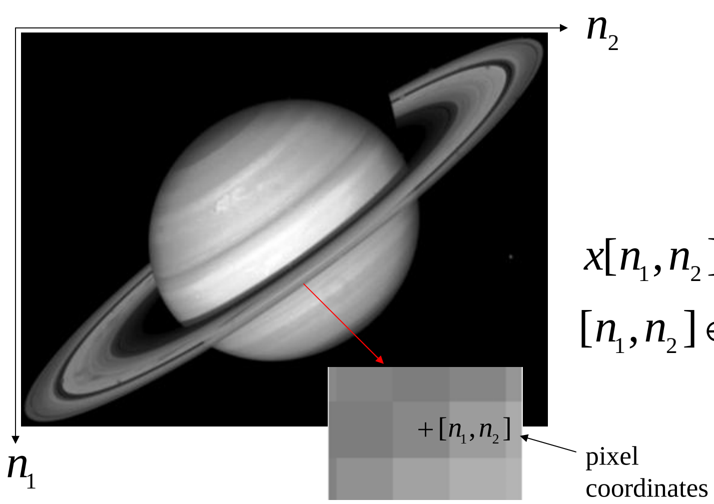


Figure 1.2 System with p inputs and q outputs.

Example of 1-D, Real-Valued, Digital Signal: Digital Audio Signal



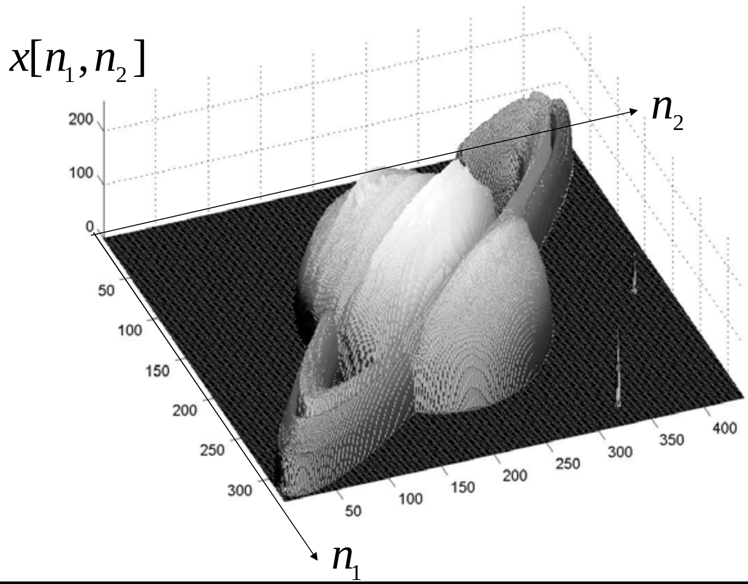
Example of 1-D, Real-Valued, Digital Signal with a 2-D Domain: A Digital Gray-Level Image



$x[n_1, n_2]$

$[n_1, n_2] \in \mathbb{Z}^2$

Digital Gray-Level Image: Cont'd

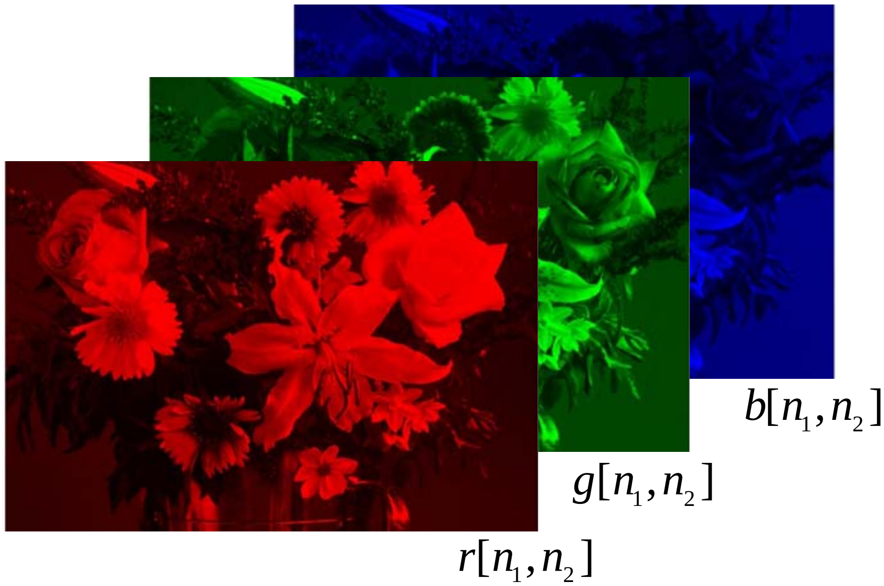


Example of 3-D, Real-Valued, Digital Signal with a 2-D Domain: A Digital Color Image

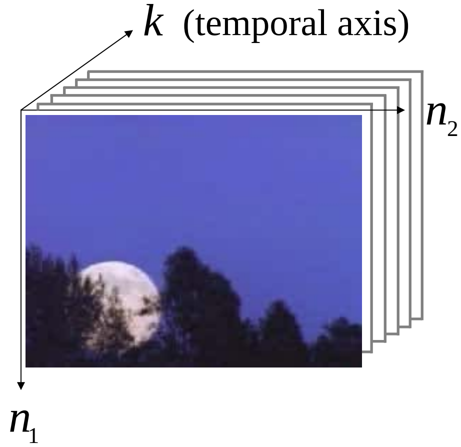
$$x[n_1, n_2] = \begin{bmatrix} r[n_1, n_2] \\ g[n_1, n_2] \\ b[n_1, n_2] \end{bmatrix}$$

$$[n_1, n_2] \in \mathbb{Z}^2$$

Digital Color Image: Cont'd



Example of 3-D, Real-Valued, Digital Signal with a 3-D Domain: A Digital Color Video Sequence



$$x[n_1,n_2,k]=\begin{bmatrix} r[n_1,n_2,k] \\ g[n_1,n_2,k] \\ b[n_1,n_2,k] \end{bmatrix}$$

$$[n_1,n_2]\in\mathbb{Z}^2, k\in\mathbb{Z}$$

Types of input/output representations considered

- Differential equation (or difference equation)
- The convolution model
- The transfer function representation (Fourier transform representation)

Examples of 1-D, Real-Valued, CT Signals: Temporal Evolution of Currents and Voltages in Electrical Circuits

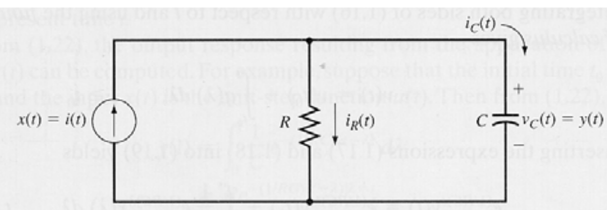


Figure 1.24 *RC* circuit.

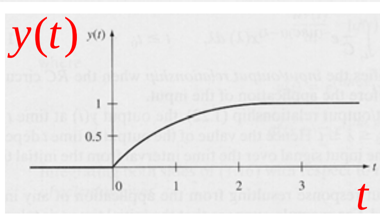


Figure 1.25 Step response of *RC* circuit when $R = C = 1$.

Examples of 1-D, Real-Valued, CT Signals: Temporal Evolution of Some Physical Quantities in Mechanical Systems

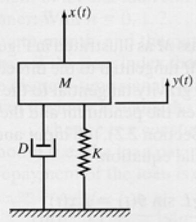
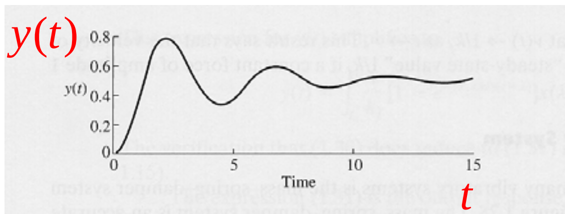


Figure 1.28 Schematic diagram of a mass–spring–damper system.



Continuous-Time (CT) Signals

- Unit-step function $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$
- Unit-ramp function $r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$

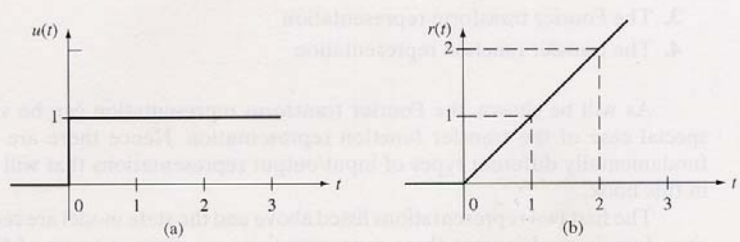


Figure 1.3 (a) Unit-step and (b) unit-ramp functions.

Unit-Ramp and Unit-Step Functions: Some Properties

$$x(t)u(t) = \begin{cases} x(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$r(t) = \int_{-\infty}^{t} u(\lambda)\, d\lambda$$

$$u(t) = \frac{dr(t)}{dt} \quad \text{(to the exception of } t = 0\text{)}$$

The Unit Impulse

- A.k.a. the delta function or Dirac distribution
- It is defined by:

$$\delta(t) = 0, \quad t \neq 0$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(\lambda)\, d\lambda = 1, \quad \forall \varepsilon > 0$$

- The value $\delta(0)$ is not defined, in particular $\delta(0) \neq \infty$

The Unit Impulse: Graphical Interpretation

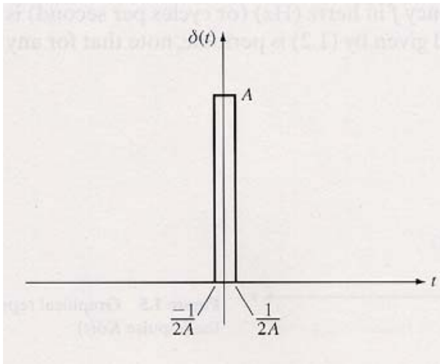


Figure 1.4 Pulse interpretation of $\delta(t)$.

A is a very large number

The Scaled Impulse $K\delta(t)$

- If $K \in \mathbb{R}$, $K\delta(t)$ is the impulse with area K, i.e.,

$$K(t) = 0, \quad t \neq 0$$

$$\int_{-\varepsilon}^{\varepsilon} K\delta(\lambda)d\lambda = K, \quad \forall \varepsilon > 0$$

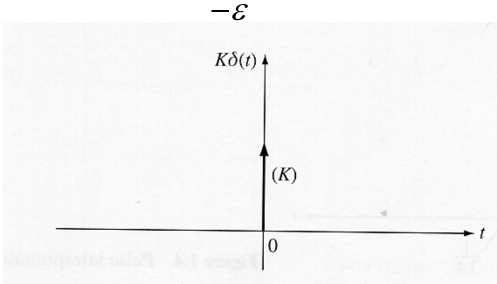


Figure 1.5 Graphical representation of the impulse $K\delta(t)$.

Properties of the Delta Function

1) $$u(t) = \int_{-\infty}^{t} \delta(\lambda)d\lambda$$

$$\forall t \text{ except } t = 0$$

2) $$\int_{t_0-\varepsilon}^{t_0+\varepsilon} x(t)\delta(t-t_0)dt = x(t_0) \quad \forall \varepsilon > 0$$

(sifting property)

Periodic Signals

- Definition: a signal $x(t)$ is said to be periodic with period T, if

$$x(t+T) = x(t) \quad \forall t \in \mathbb{R}$$

- Notice that $x(t)$ is also periodic with period qT where q is any positive integer
- T is called the fundamental period

Example: The Sinusoid

$$x(t) = A\cos(\omega t + \theta), \quad t \in \mathbb{R}$$

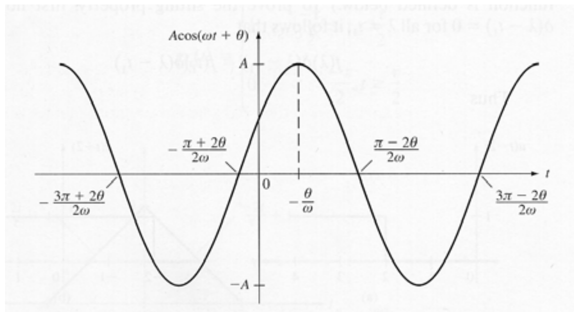


Figure 1.6 Sinusoid $A\cos(\omega t + \theta)$ with $-\pi/2 < \theta < 0$.

ω [rad / sec]

θ [rad]

$$f = \frac{\omega}{2\pi} \; [1/\sec] = [Hz]$$

Is the Sum of Periodic Signals Periodic?

- Let $x_1(t)$ and $x_2(t)$ be two periodic signals with periods T_1 and T_2, respectively
- Then, the sum $x_1(t) + x_2(t)$ is periodic only if the ratio T_1/T_2 can be written as the ratio q/r of two integers q and r
- In addition, if r and q are coprime, then $T = rT_1$ is the fundamental period of $x_1(t) + x_2(t)$

Time-Shifted Signals

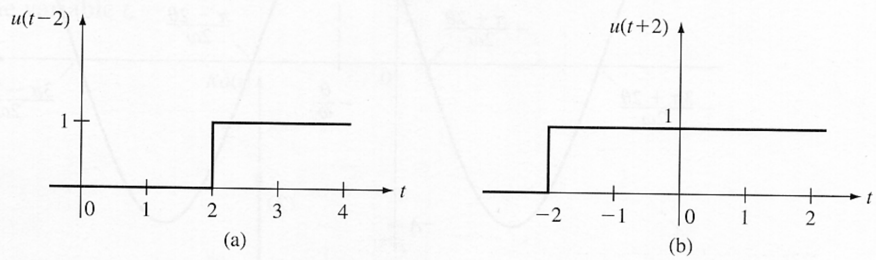
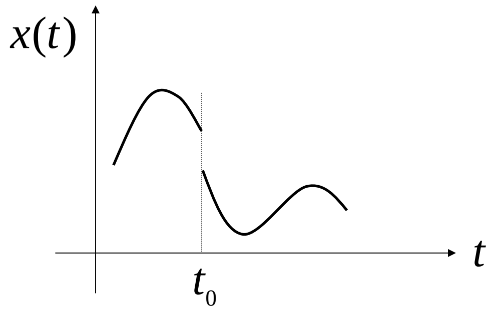


Figure 1.7 Two-second shifts of $u(t)$: (a) right shift; (b) left shift.

Points of Discontinuity

- A continuous-time signal $x(t)$ is said to be discontinuous at a point t_0 if $x(t_0^+) \neq x(t_0^-)$ where $t_0^+ = t_0 + \varepsilon$ and $t_0^- = t_0 - \varepsilon$, ε being a small positive number



Continuous Signals

- A signal $x(t)$ is continuous at the point t_0 if $x(t_0^+) = x(t_0^-)$

- If a signal $x(t)$ is continuous at all points t, $x(t)$ is said to be a continuous signal

Example of Continuous Signal: The Triangular Pulse Function

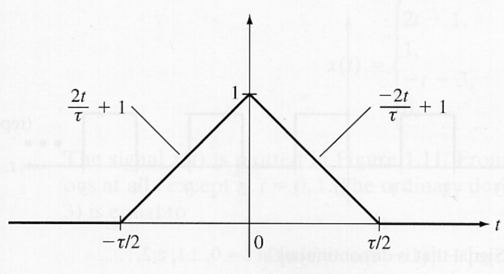


Figure 1.8 Triangular pulse function.

Piecewise-Continuous Signals

- A signal $x(t)$ is said to be piecewise continuous if it is continuous at all t except a finite or countably infinite collection of points $t_i,\ i = 1, 2, 3, \ldots$

Example of Piecewise-Continuous Signal: The Rectangular Pulse Function

$$p_\tau(t) = u(t + \tau/2) - u(t - \tau/2)$$

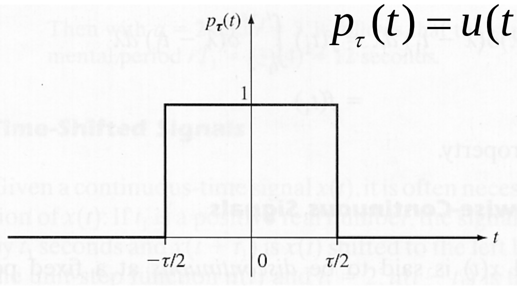


Figure 1.9 Rectangular pulse function.

Another Example of Piecewise-Continuous Signal: The Pulse Train Function

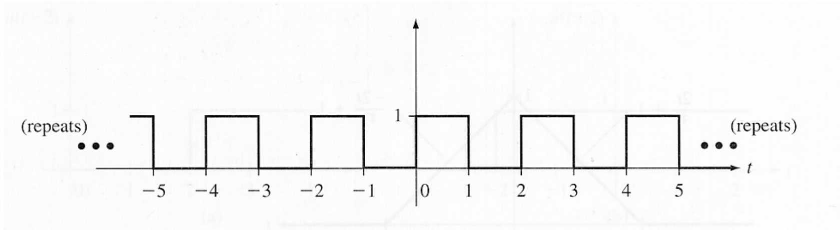


Figure 1.10 Signal that is discontinuous at $t = 0, \pm 1, \pm 2, \ldots$

Derivative of a Continuous-Time Signal

- A signal $x(t)$ is said to be differentiable at a point t_0 if the quantity

$$\frac{x(t_0 + h) - x(t_0)}{h}$$

  has limit as $h \to 0$ independent of whether h approaches 0 from above $(h > 0)$ or from below $(h < 0)$

- If the limit exists, $x(t)$ has a derivative at t_0

$$\left.\frac{dx(t)}{dt}\right|_{t=t_0} = \lim_{h \to 0} \frac{x(t_0 + h) - x(t_0)}{h}$$

Continuity and Differentiability

- In order for $x(t)$ to be differentiable at a point t_0, it is necessary (but not sufficient) that $x(t)$ be continuous at t_0

- Continuous-time signals that are not continuous at all points (piecewise continuity) cannot be differentiable at all points

Generalized Derivative

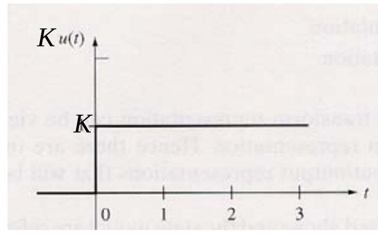
- However, piecewise-continuous signals may have a derivative in a generalized sense
- Suppose that $x(t)$ is differentiable at all t except $t = t_0$
- The generalized derivative of $x(t)$ is defined to be

$$\frac{dx(t)}{dt} + \left[x(t_0^+) - x(t_0^-) \right] \delta(t - t_0)$$

ordinary derivative of $x(t)$ at all t except $t = t_0$

Example: Generalized Derivative of the Step Function

- Define $x(t) = Ku(t)$



- The ordinary derivative of $x(t)$ is 0 at all points except $t = 0$
- Therefore, the generalized derivative of $x(t)$ is

$$K\left[u(0^+) - u(0^-)\right]\delta(t-0) = K\delta(t)$$

Another Example of Generalized Derivative

- Consider the function defined as

$$x(t) = \begin{cases} 2t+1, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ -t+3, & 2 \le t \le 3 \\ 0, & \textit{all other } t \end{cases}$$

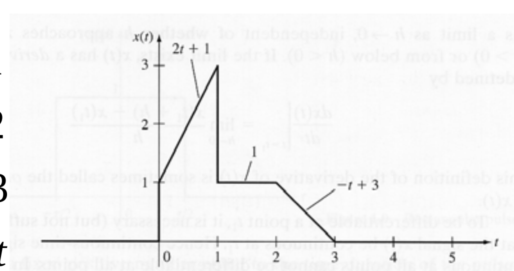


Figure 1.11 Signal in Example 1.3.

Another Example of Generalized Derivative: Cont'd

- The ordinary derivative of $x(t)$, at all t except $t = 0,1,2,3$ is

$$\frac{dx(t)}{dt} = 2\left[u(t) - u(t-1)\right] - \left[u(t-2) - u(t-3)\right]$$

- Its generalized derivative is

$$\frac{dx(t)}{dt} + \underbrace{\left[x(0^+) - x(0^-)\right]}_{1}\delta(t) + \underbrace{\left[x(1^+) - x(1^-)\right]}_{-2}\delta(t-1)$$

Another Example of Generalized Derivative: Cont'd

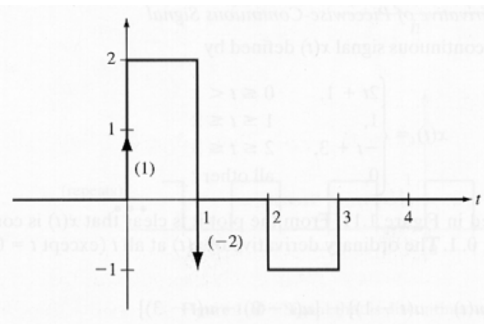


Figure 1.12 Generalized derivative of the signal in Example 1.3.

Signals Defined Interval by Interval

- Consider the signal

$$x(t) = \begin{cases} x_1(t), & t_1 \le t < t_2 \\ x_2(t), & t_2 \le t < t_3 \\ x_3(t), & t \ge t_3 \end{cases}$$

- This signal can be expressed in terms of the unit-step function $u(t)$ and its time-shifts as

$$\begin{aligned} x(t) = x_1(t)\left[u(t-t_1) - u(t-t_2)\right] + \\ + x_2(t)\left[u(t-t_2) - u(t-t_3)\right] + \\ + x_3(t)u(t-t_3), \quad t \ge t_1 \end{aligned}$$

Signals Defined Interval by Interval: Cont'd

- By rearranging the terms, we can write

$$x(t) = f_1(t)u(t-t_1) + f_2(t)u(t-t_2) + f_3(t)u(t-t_3)$$

where

$$f_1(t) = x_1(t)$$
$$f_2(t) = x_2(t) - x_1(t)$$
$$f_3(t) = x_3(t) - x_2(t)$$

Discrete-Time (DT) Signals

- A discrete-time signal is defined only over integer values
- We denote such a signal by

$$x[n], \ \ n \in \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

Example: A Discrete-Time Signal Plotted with Matlab

- Suppose that

$$x[0]=1, \quad x[1]=2, \quad x[2]=1, \quad x[3]=0, \quad x[4]=-1$$

```
n=-2:6;
x=[0 0 1 2 1 0 –1 0 0];
stem(n,x)
xlabel('n')
ylabel('x[n]')
```

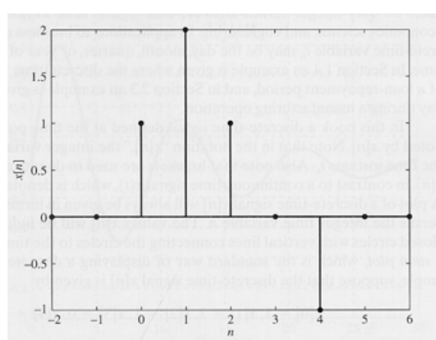
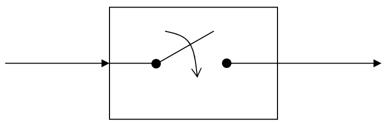


Figure 1.15 MATLAB plot of x[n].

Sampling

- Discrete-time signals are usually obtained by sampling continuous-time signals

$x(t)$



$$x[n] = x(t)\big|_{t=nT} = x(nT)$$

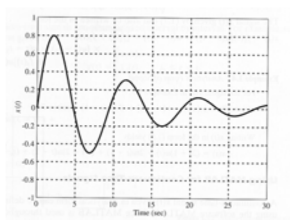


Figure 1.13 MATLAB plot of the signal $x(t) = e^{-0.1t}\sin\frac{2}{3}t$.

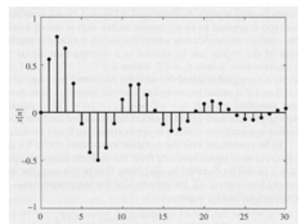


Figure 1.17 Sampled continuous-time signal.

DT Step and Ramp Functions

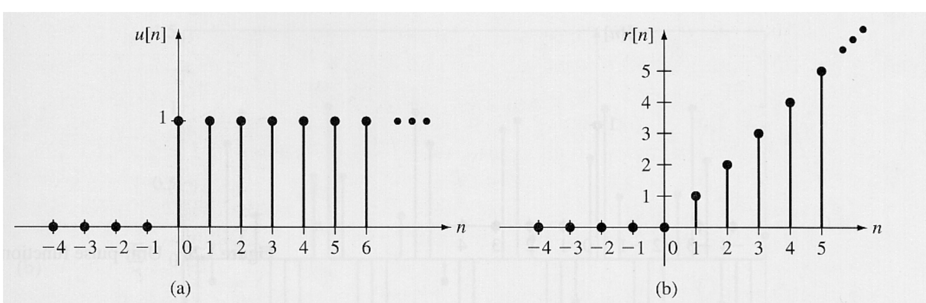


Figure 1.18 (a) Discrete-time unit step and (b) unit-ramp functions.

DT Unit Pulse

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

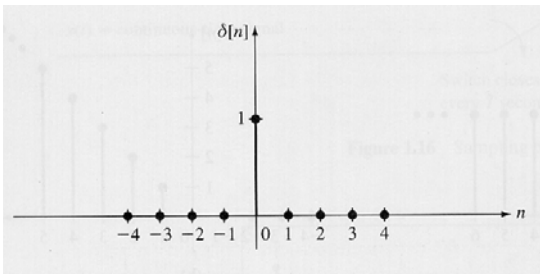


Figure 1.20 Unit-pulse function.

Periodic DT Signals

- A DT signal $x[n]$ is periodic if there exists a positive integer r such that

$$x[n+r] = x[n] \quad \forall n \in \mathbb{Z}$$

- r is called the period of the signal
- The fundamental period is the smallest value of r for which the signal repeats

Example: Periodic DT Signals

- Consider the signal $x[n] = A\cos(\Omega n + \theta)$
- The signal is periodic if

$$A\cos(\Omega(n+r)+\theta) = A\cos(\Omega n + \theta)$$

- Recalling the periodicity of the cosine

$$\cos(\alpha) = \cos(\alpha + 2k\pi)$$

$x[n]$ is periodic if and only if there exists a positive integer r such that $\Omega r = 2k\pi$ for some integer k or, equivalently, that the DT frequency Ω is such that $\Omega = 2k\pi / r$ for some positive integers k and r

Example: $x[n] = A\cos(\Omega n + \theta)$ for different values of Ω

$\Omega = \pi/3, \theta = 0$

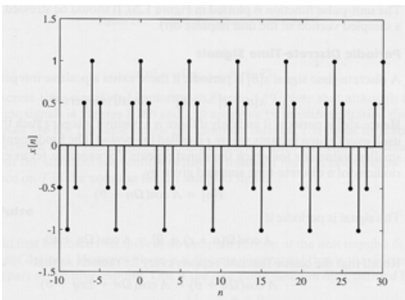


Figure 1.21 Discrete-time sinusoids with $\theta = 0$ and (a) $\Omega = \pi/3$ and (b) $\Omega = 1$.

periodic signal with period $r = 6$

$\Omega = 1, \theta = 0$

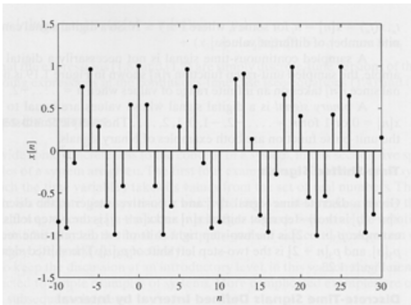


Figure 1.21 (continued)

aperiodic signal
(with periodic envelope)

DT Rectangular Pulse

$$p_L[n] = \begin{cases} 1, & n = -(L-1)/2, \ldots, -1, 0, 1, \ldots, (L-1)/2 \\ 0, & \textit{all other } n \end{cases}$$

(L must be an odd integer)

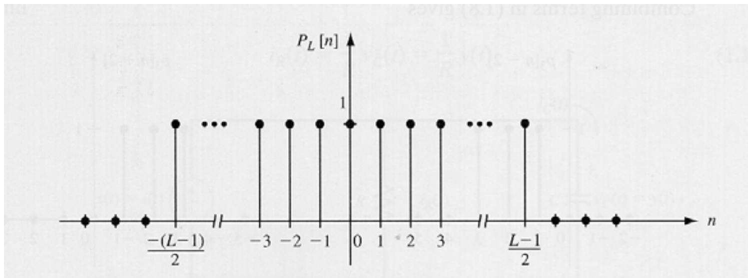


Figure 1.22 Discrete-time rectangular pulse.

Digital Signals

- A digital signal $x[n]$ is a DT signal whose values belong to a finite set or alphabet $\{a_1, a_2, \ldots, a_N\}$
- A CT signal can be converted into a digital signal by cascading the ideal sampler with a quantizer

Time-Shifted Signals

- If $x[n]$ is a DT signal and q is a positive integer

$x[n-q]$ is the q-step right shift of $x[n]$

$x[n+q]$ is the q-step left shift of $x[n]$

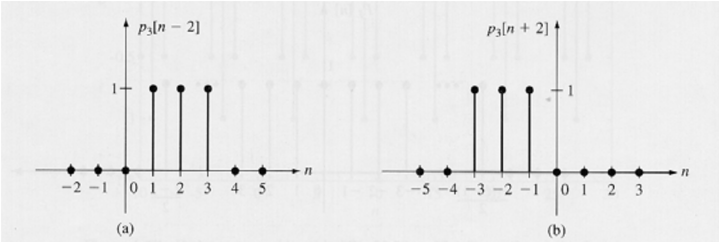


Figure 1.23 Two-step shifts of $p_3[n]$: (a) right shift; (b) left shift.

Example of CT System: An RC Circuit

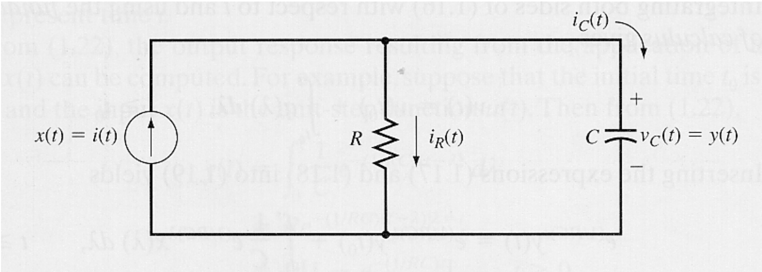


Figure 1.24 *RC* circuit.

Kirchhoff's current law: $i_C(t) + i_R(t) = i(t)$

RC Circuit: Cont'd

- The *v*-*i* law for the capacitor is

$$i_C(t) = C\frac{dv_C(t)}{dt} = C\frac{dy(t)}{dt}$$

- Whereas for the resistor it is

$$i_R(t) = \frac{1}{R}v_C(t) = \frac{1}{R}y(t)$$

RC Circuit: Cont'd

- Constant-coefficient linear differential equation describing the I/O relationship if the circuit

$$C\frac{dy(t)}{dt} + \frac{1}{R}y(t) = i(t) = x(t)$$

RC Circuit: Cont'd

- Step response when R=C=1

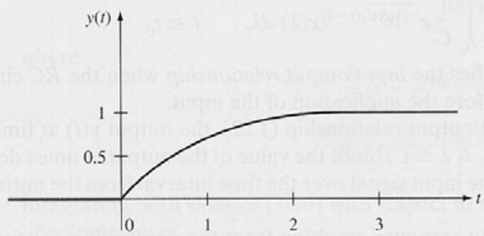


Figure 1.25 Step response of RC circuit when $R = C = 1$.

Example of CT System: Car on a Level Surface

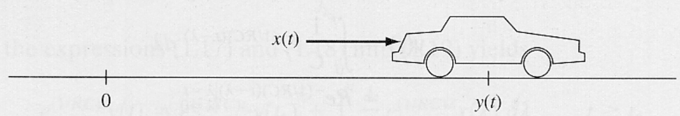


Figure 1.26 Car with drive or braking force $x(t)$.

Newton's second law of motion:

$$M\frac{d^2y(t)}{dt^2}+k_f\frac{dy(t)}{dt}=x(t)$$

where $x(t)$ is the drive or braking force applied to the car at time t and $y(t)$ is the car's position at time t

Car on a Level Surface: Cont'd

- Step response when M=1 and $k_f = 0.1$

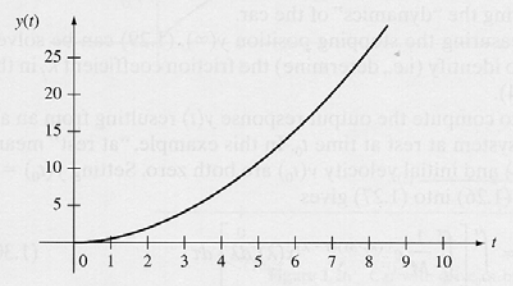


Figure 1.27 Step response of car with $M = 1$ and $k_f = 0.1$.

Example of CT System: Mass-Spring-Damper System

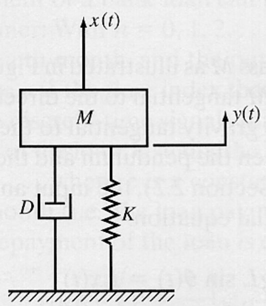
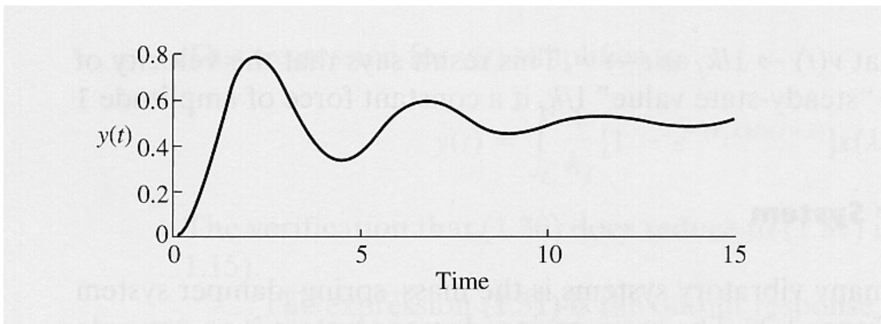


Figure 1.28 Schematic diagram of a mass–spring–damper system.

$$M\frac{d^2y(t)}{dt^2} + D\frac{dy(t)}{dt} + Ky(t) = x(t)$$

Mass-Spring-Damper System: Cont'd

- Step response when M=1, K=2, and D=0.5



Example of CT System: Simple Pendulum

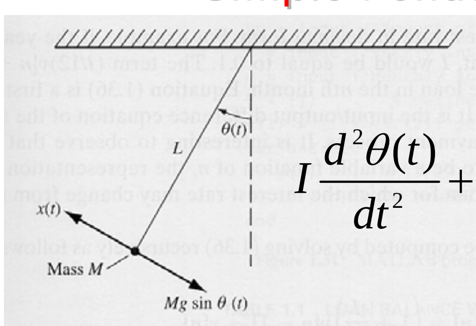


Figure 1.30 Simple pendulum.

$$I\frac{d^2\theta(t)}{dt^2} + MgL\sin\theta(t) = Lx(t)$$

If $\sin\theta(t) \approx \theta(t)$

$$I\frac{d^2\theta(t)}{dt^2} + MgL\,\theta(t) = Lx(t)$$

Basic System Properties: Causality

- A system is said to be causal if, for any time t_1, the output response at time t_1 resulting from input $x(t)$ does not depend on values of the input for $t > t_1$.
- A system is said to be noncausal if it is not causal

Example: The Ideal Predictor

$$y(t) = x(t+1)$$

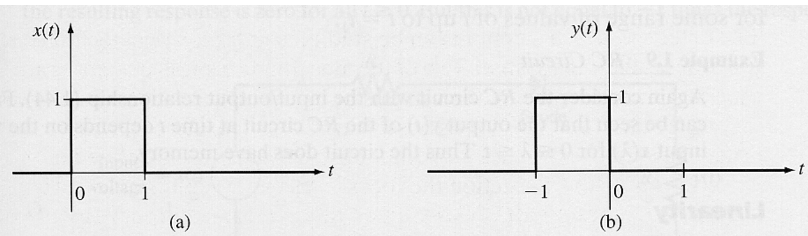


Figure 1.32 (a) Input and (b) output pulse of system in Example 1.5.

Example: The Ideal Delay

$$y(t) = x(t-1)$$

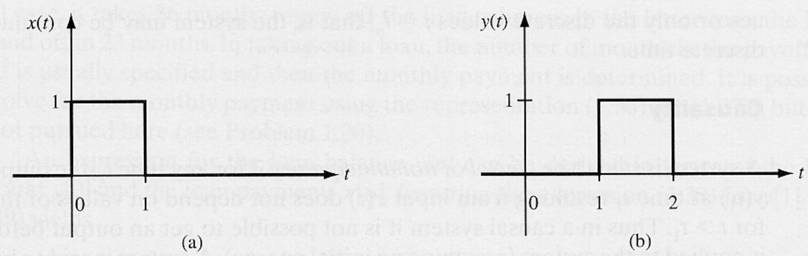


Figure 1.33 (a) Input and (b) output pulse of system in Example 1.6.

Memoryless Systems and Systems with Memory

- A causal system is memoryless or static if, for any time t_1, the value of the output at time t_1 depends only on the value of the input at time t_1
- A causal system that is not memoryless is said to have memory. A system has memory if the output at time t_1 depends in general on the past values of the input $x(t)$ for some range of values of t up to $t = t_1$

Examples

- Ideal Amplifier/Attenuator

$$y(t) = Kx(t)$$

- RC Circuit

$$y(t) = \frac{1}{C}\int_0^t e^{-(1/RC)(t-\tau)}x(\tau)d\tau, \quad t \geq 0$$

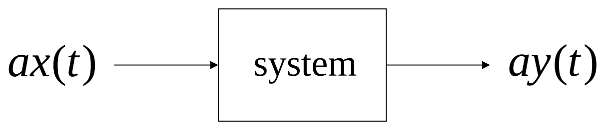
Basic System Properties: Additive Systems

- A system is said to be additive if, for any two inputs $x_1(t)$ and $x_2(t)$, the response to the sum of inputs $x_1(t) + x_2(t)$ is equal to the sum of the responses to the inputs, assuming no initial energy before the application of the inputs

$$x_1(t) + x_2(t) \longrightarrow \boxed{\text{system}} \longrightarrow y_1(t) + y_2(t)$$

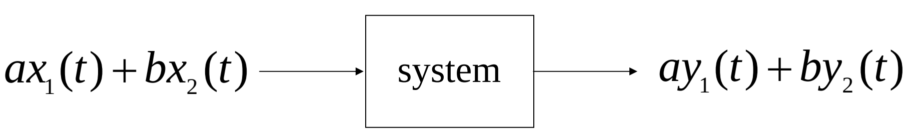
Basic System Properties: Homogeneous Systems

- A system is said to be homogeneous if, for any input $x(t)$ and any scalar a, the response to the input $ax(t)$ is equal to a times the response to $x(t)$, assuming no energy before the application of the input



Basic System Properties: Linearity

- A system is said to be linear if it is both additive and homogeneous



- A system that is not linear is said to be nonlinear

Example of Nonlinear System: Circuit with a Diode

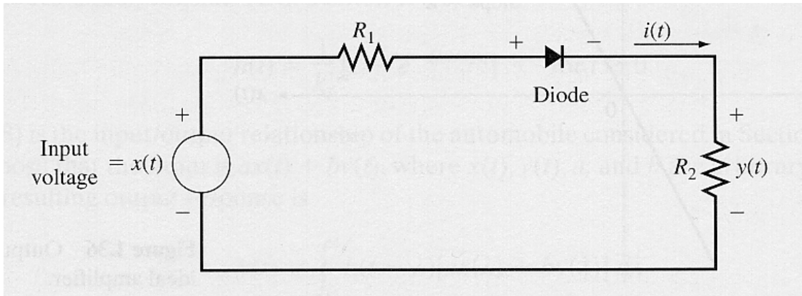


Figure 1.34 Resistive circuit with an ideal diode.

$$
y(t) = \begin{cases} \dfrac{R_2}{R_1 + R_2} x(t), & when\ x(t) \geq 0 \\ 0, & when\ x(t) \leq 0 \end{cases}
$$

Example of Nonlinear System: Square-Law Device

$$
y(t) = x^2(t)
$$

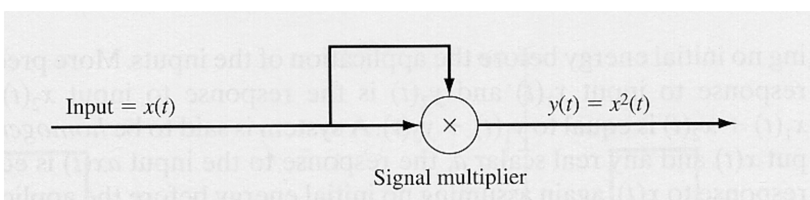


Figure 1.35 Realization of $y(t) = x^2(t)$.

Example of Linear System: The Ideal Amplifier

$$y(t) = Kx(t)$$

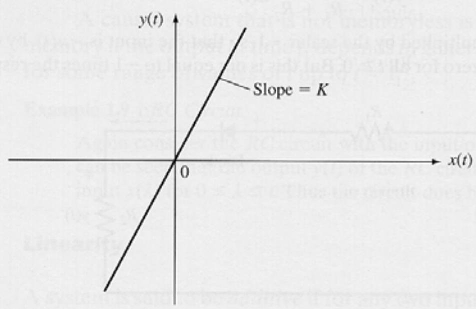


Figure 1.36 Output versus input in an ideal amplifier.

Example of Linear System: A Real Amplifier

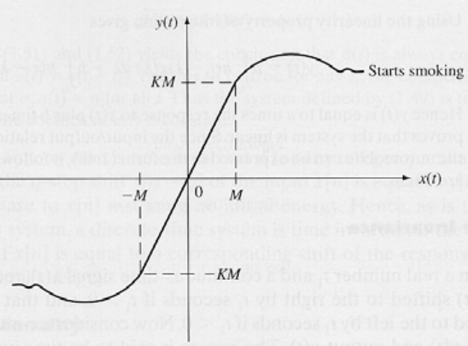


Figure 1.37 Output versus input in a nonideal amplifier.

Basic System Properties: Time Invariance

- A system is said to be time invariant if, for any input $x(t)$ and any time t_1, the response to the shifted input $x(t-t_1)$ is equal to $y(t-t_1)$ where $y(t)$ is the response to $x(t)$ with zero initial energy

$$x(t-t_1) \longrightarrow \boxed{\text{system}} \longrightarrow y(t-t_1)$$

- A system that is not time invariant is said to be time varying or time variant

Examples of Time Varying Systems

- Amplifier with Time-Varying Gain

$$y(t) = t x(t)$$

- First-Order System

$$\dot{y}(t) + a(t) y(t) = b x(t)$$

Basic System Properties: Finite Dimensionality

- Let x(t) and y(t) be the input and output of a CT system
- Let $x^{(i)}(t)$ and $y^{(i)}(t)$ denote their i-th derivatives
- The system is said to be finite dimensional or lumped if, for some positive integer N the N-th derivative of the output at time t is equal to a function of $x^{(i)}(t)$ and $y^{(i)}(t)$ at time t for $0 \le i \le N-1$

Basic System Properties: Finite Dimensionality – Cont'd

- The N-th derivative of the output at time t may also depend on the i-th derivative of the input at time t for $i \ge N$

$$y^{(N)}(t) = f(y(t), y^{(1)}(t), \ldots, y^{(N-1)}(t),$$
$$x(t), x^{(1)}(t), \ldots, x^{(M)}(t), t)$$

- The integer N is called the order of the above I/O differential equation as well as the order or dimension of the system described by such equation

Basic System Properties: Finite Dimensionality – Cont'd

- A CT system with memory is infinite dimensional if it is not finite dimensional, *i.e.*, if it is not possible to express the N-th derivative of the output in the form indicated above for some positive integer N
- Example: System with Delay

$$\frac{dy(t)}{dt} + ay(t-1) = x(t)$$

DT Finite-Dimensional Systems

- Let $x[n]$ and $y[n]$ be the input and output of a DT system.
- The system is finite dimensional if, for some positive integer N and nonnegative integer M, $y[n]$ can be written in the form

$$y[n] = f\,(y[n-1], y[n-2], \ldots, y[n-N], \\ x[n], x[n-1], \ldots, x[n-M], n)$$

- N is called the order of the I/O difference equation as well as the order or dimension of the system described by such equation

Basic System Properties: CT Linear Finite-Dimensional Systems

- If the N-th derivative of a CT system can be written in the form

$$y^{(N)}(t) = -\sum_{i=0}^{N-1} a_i(t) y^{(i)}(t) + \sum_{i=0}^{M} b_i(t) x^{(i)}(t)$$

then the system is both linear and finite dimensional

Basic System Properties: DT Linear Finite-Dimensional Systems

- If the output of a DT system can be written in the form

$$y[n] = -\sum_{i=0}^{N-1} a_i(n) y[n-i] + \sum_{i=0}^{M} b_i(n) x[n-i]$$

then the system is both linear and finite dimensional

Basic System Properties: Linear Time-Invariant Finite-Dimensional Systems

- For a CT system it must be

$$a_i(t) = a_i \quad and \quad b_i(t) = b_i \quad \forall\ i\ and\ t \in \mathbb{R}$$

- And, similarly, for a DT system

$$a_i(n) = a_i \quad and \quad b_i(n) = b_i \quad \forall\ i\ and\ n \in \mathbb{Z}$$

About the Order of Differential and Difference Equations

✓ • Some authors define the order as N

- Some as (M, N)
- Some others as $\max(M, N)$