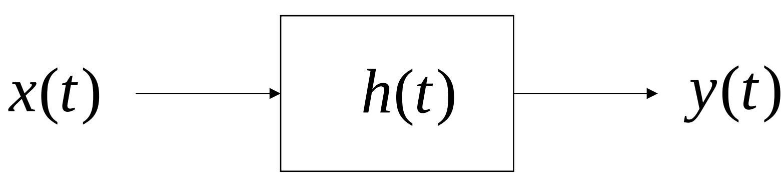
Chapter 5
Frequency Domain Analysis of Systems

CT, LTI Systems

- Consider the following CT LTI system:



- Assumption: the impulse response $h(t)$ is *absolutely integrable,* i.e.,

$$\int_{\mathbb{R}} |\, h(t) \,|\, dt < \infty$$

(this has to do with system stability)

Response of a CT, LTI System to a Sinusoidal Input

- What's the response $y(t)$ of this system to the input signal

$$x(t) = \cos(\omega_0 t + \theta), \ \ t \in \mathbb{R} \quad ?$$

- We start by looking for the response $y_c(t)$ of the same system to

$$x_c(t) = e^{j\omega_0 t} \quad t \in \mathbb{R}$$

Response of a CT, LTI System to a Complex Exponential Input

- The output is obtained through convolution as

$$y_c(t) = h(t) * x_c(t) = \int_{\mathbb{R}} h(\tau) x_c(t-\tau) d\tau =$$

$$= \int_{\mathbb{R}} h(\tau) e^{j\omega_0 (t-\tau)} d\tau =$$

$$= \underbrace{e^{j\omega_0 t}}_{x_c(t)} \int_{\mathbb{R}} h(\tau) e^{-j\omega_0 \tau} d\tau =$$

$$= x_c(t) \int_{\mathbb{R}} h(\tau) e^{-j\omega_0 \tau} d\tau$$

The Frequency Response of a CT, LTI System

- By defining

$$H(\omega) = \int_{\mathbb{R}} h(\tau) e^{-j\omega\tau} d\tau$$

$H(\omega)$ is the frequency response of the CT, LTI system = Fourier transform of $h(t)$

it is

$$y_c(t) = H(\omega_0) x_c(t) =$$

$$= H(\omega_0) e^{j\omega_0 t}, \quad t \in \mathbb{R}$$

- Therefore, the response of the LTI system to a complex exponential is another complex exponential with the same frequency ω_0

Analyzing the Output Signal $y_c(t)$

- Since $H(\omega_0)$ is in general a complex quantity, we can write

$$y_c(t) = H(\omega_0) e^{j\omega_0 t} =$$

$$= |H(\omega_0)| e^{j \arg H(\omega_0)} e^{j\omega_0 t} =$$

$$= \underbrace{|H(\omega_0)|}_{\text{output signal's magnitude}} e^{j(\omega_0 t + \underbrace{\arg H(\omega_0)}_{\text{output signal's phase}})}$$

Response of a CT, LTI System to a Sinusoidal Input

- With Euler's formulas we can express x(t) as

$$x(t) = \cos(\omega_0 t + \theta)$$
$$= \tfrac{1}{2}(e^{j(\omega_0 t + \theta)} + e^{-j(\omega_0 t + \theta)})$$
$$= \tfrac{1}{2}e^{j\theta}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\theta}e^{-j\omega_0 t}$$

Using the previous result, the response is

$$y(t) = \tfrac{1}{2}e^{j\theta}H(\omega_0)e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\theta}H(-\omega_0)e^{-j\omega_0 t}$$

Response of a CT, LTI System to a Sinusoidal Input – Cont'd

- If h(t) is real, then $H(-\omega) = H^*(\omega)$ and

$$H(\omega_0) = |H(\omega_0)|e^{j\arg H(\omega_0)}$$
$$H(-\omega_0) = |H(\omega_0)|e^{-j\arg H(\omega_0)}$$

- Thus we can write y(t) as

$$y(t) = \frac{1}{2}|H(\omega_0)|e^{j(\omega_0 t + \theta + \arg H(\omega_0))} + \frac{1}{2}|H(\omega_0)|e^{-j(\omega_0 t + \theta + \arg H(\omega_0))}$$
$$= |H(\omega_0)|\cos(\omega_0 t + \theta + \arg H(\omega_0))$$

Response of a CT, LTI System to a Sinusoidal Input – Cont'd

- Thus, the response to

$$x(t) = A\cos(\omega_0 t + \theta)$$

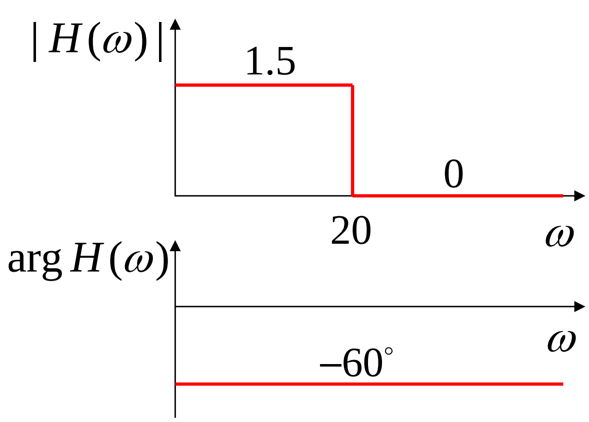
is

$$y(t) = A\,|H(\omega_0)|\cos\big(\omega_0 t + \theta + \arg H(\omega_0)\big)$$

which is also a sinusoid with the same frequency ω_0 but with the amplitude scaled by the factor $|H(\omega_0)|$ and with the phase shifted by amount $\arg H(\omega_0)$

Example: Response of a CT, LTI System to Sinusoidal Inputs

- Suppose that the frequency response of a CT, LTI system is defined by the following specs:



$$|H(\omega)| = \begin{cases} 1.5, & 0 \le \omega \le 20, \\ 0, & \omega > 20, \end{cases}$$

$$\arg H(\omega) = -60^\circ, \forall \omega$$

Example: Response of a CT, LTI System to Sinusoidal Inputs – Cont'd

- If the input to the system is

$$x(t) = 2\cos(10t + 90^\circ) + 5\cos(25t + 120^\circ)$$

- Then the output is

$$y(t) = 2\,|H(10)|\cos(10t + 90^\circ + \arg H(10)) +$$
$$+ 5\,|H(25)|\cos(25t + 120^\circ + \arg H(25)) =$$
$$= 3\cos(10t + 30^\circ)$$

Example: Frequency Analysis of an RC Circuit

- Consider the RC circuit shown in figure

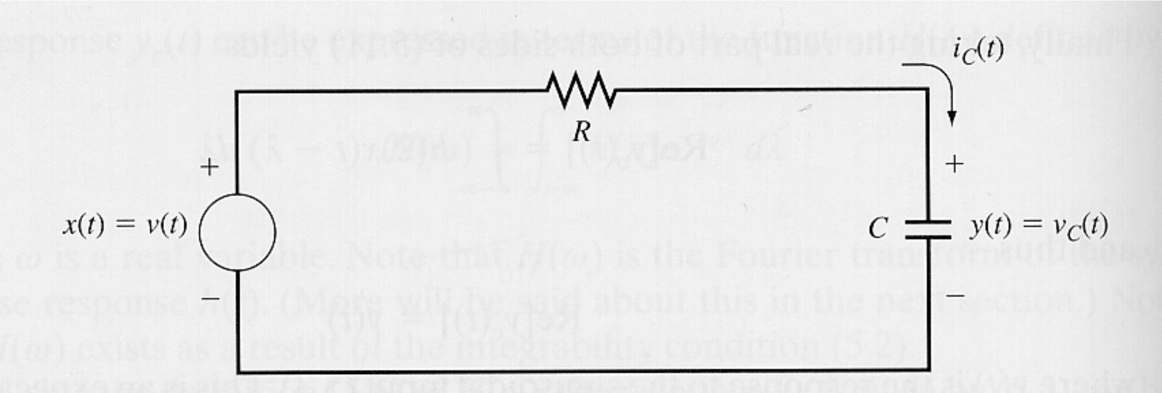


Figure 5.1 *RC* circuit in Example 5.2.

Example: Frequency Analysis of an RC Circuit – Cont'd

- From EEE2032F, we know that:
  1. The complex impedance of the capacitor is equal to $1/j\omega C$
  2. If the input voltage is $x_c(t) = e^{j\omega t}$, then the output signal is given by

$$y_c(t) = \frac{1/j\omega C}{R + 1/j\omega C} e^{j\omega t} = \frac{1/RC}{j\omega + 1/RC} e^{j\omega t}$$

Example: Frequency Analysis of an RC Circuit – Cont'd

- Setting $\omega = \omega_0$, it is

$$x_c(t) = e^{j\omega_0 t} \quad \text{and} \quad y_c(t) = \frac{1/RC}{j\omega_0 + 1/RC} e^{j\omega_0 t}$$

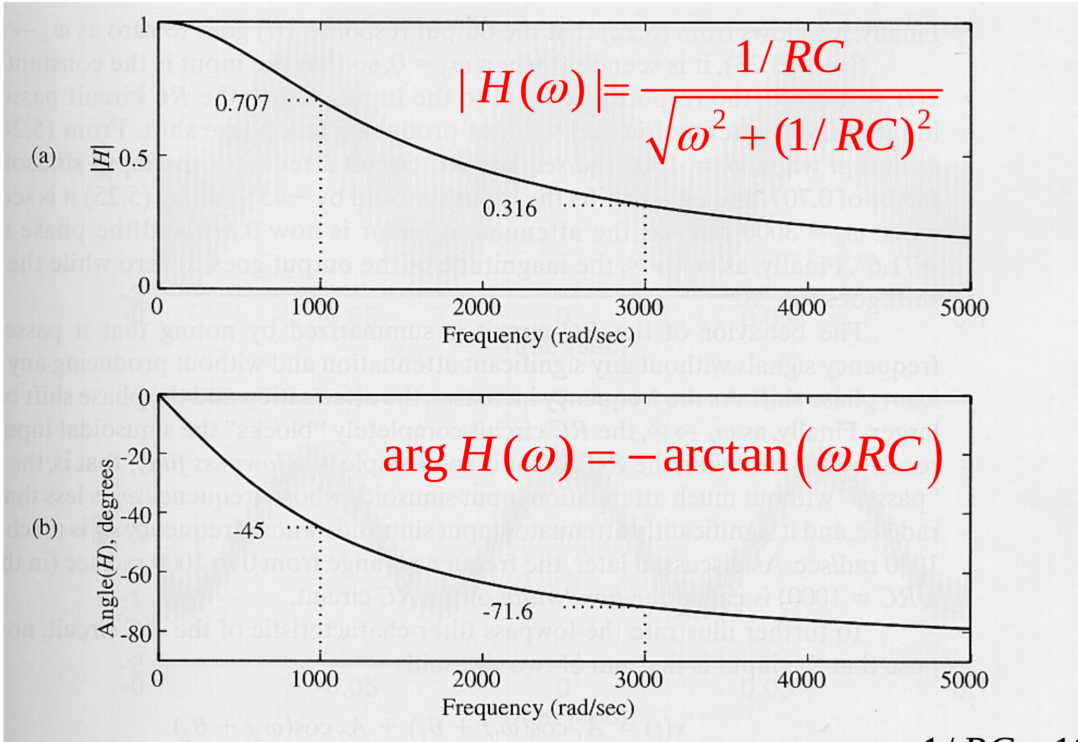
whence we can write

$$y_c(t) = H(\omega_0) x_c(t)$$

where

$$H(\omega) = \frac{1/RC}{j\omega + 1/RC}$$

Example: Frequency Analysis of an RC Circuit – Cont'd



$1/RC = 1000$

Example: Frequency Analysis of an RC Circuit – Cont'd

- The knowledge of the frequency response $H(\omega)$ allows us to compute the response $y(t)$ of the system to any sinusoidal input signal

$$x(t) = A\cos(\omega_0 t + \theta)$$

since

$$y(t) = A|H(\omega_0)|\cos\left(\omega_0 t + \theta + \arg H(\omega_0)\right)$$

Example: Frequency Analysis of an RC Circuit – Cont'd

- Suppose that $1/RC = 1000$ and that

$$x(t) = \cos(100t) + \cos(3000t)$$

- Then, the output signal is

$$y(t) = |H(100)|\cos(100t + \arg H(100)) +$$
$$+ |H(3000)|\cos(3000t + \arg H(3000)) =$$
$$= 0.9950\cos(100t - 5.71^{\circ}) + 0.3162\cos(3000t - 71.56^{\circ})$$

Example: Frequency Analysis of an RC Circuit – Cont'd

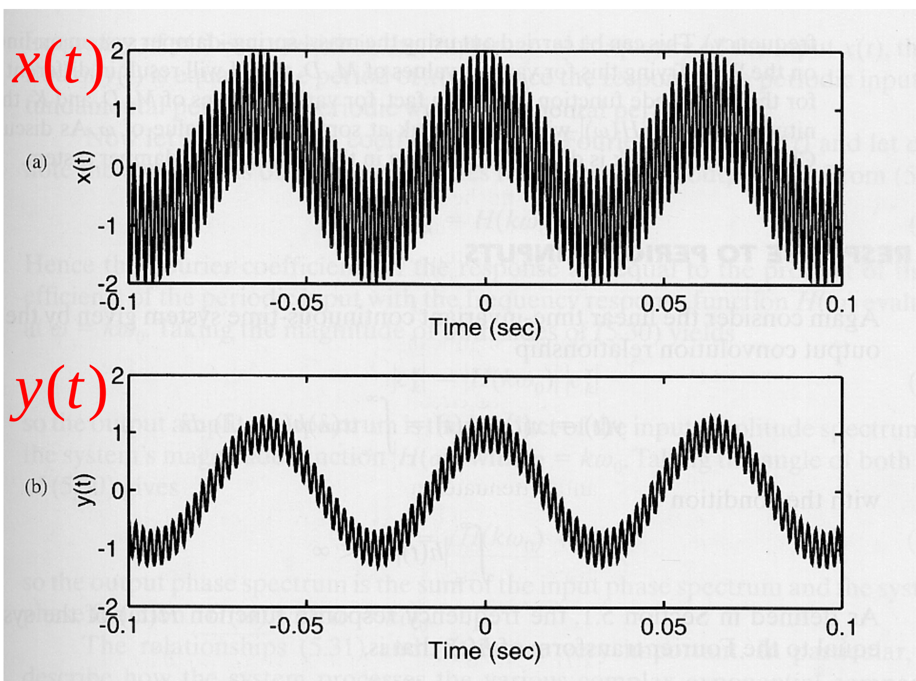


Figure 5.3 (a) Input and (b) output of *RC* circuit when $1/RC = 1000$.

Example: Frequency Analysis of an RC Circuit – Cont'd

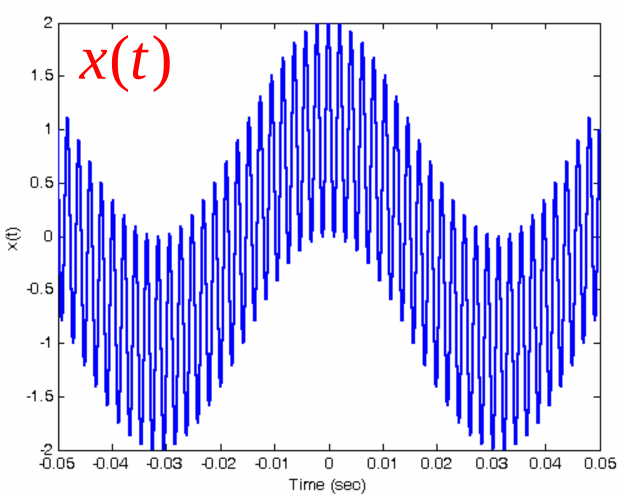
- Suppose now that

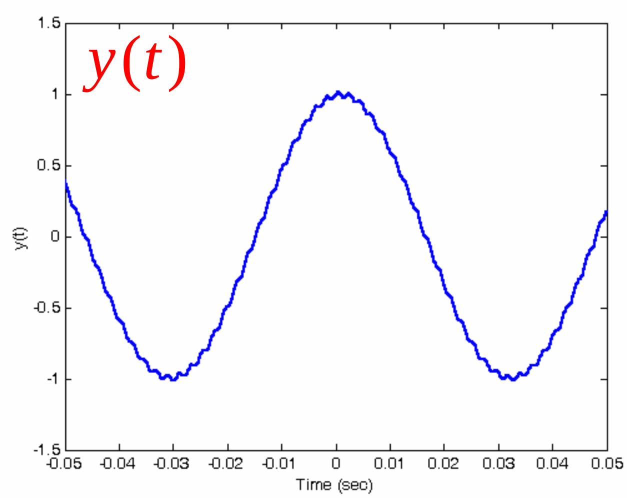
$$x(t) = \cos(100t) + \cos(50{,}000t)$$

- Then, the output signal is

$$y(t) = |H(100)|\cos(100t + \arg H(100)) +$$
$$+ |H(50{,}000)|\cos(50{,}000t + \arg H(50{,}000)) =$$
$$= 0.9950\cos(100t - 5.71^\circ) + 0.0200\cos(50{,}000t - 88.85^\circ)$$

Example: Frequency Analysis of an RC Circuit – Cont'd





The RC circuit behaves as a lowpass filter, by letting low-frequency sinusoidal signals pass with little attenuation and by significantly attenuating high-frequency sinusoidal signals

Response of a CT, LTI System to Periodic Inputs

- Suppose that the input to the CT, LTI system is a periodic signal $x(t)$ having period T
- This signal can be represented through its Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k^x e^{jk\omega_0 t}, \quad t \in \mathbb{R}$$

where

$$c_k^x = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt, \quad k \in \mathbb{Z}$$

Response of a CT, LTI System to Periodic Inputs – Cont'd

- By exploiting the previous results and the linearity of the system, the output of the system is

$$y(t) = \sum_{k=-\infty}^{\infty} H(k\omega_0) c_k^x e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{|H(k\omega_0)||c_k^x|}_{|c_k^y|} e^{j(k\omega_0 t + \underbrace{\arg(c_k^x) + \arg H(k\omega_0)}_{\arg c_k^y})} =$$

$$= \sum_{k=-\infty}^{\infty} |c_k^y| e^{j(k\omega_0 t + \arg(c_k^y))} = \sum_{k=-\infty}^{\infty} c_k^y e^{jk\omega_0 t}, \qquad t \in \mathbb{R}$$

Example: Response of an RC Circuit to a Rectangular Pulse Train

- Consider the RC circuit

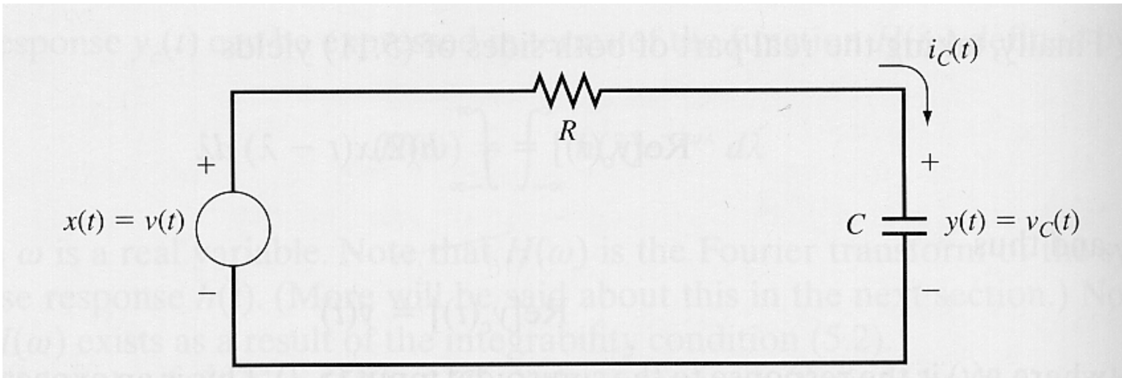


Figure 5.1 *RC* circuit in Example 5.2.

with input $$x(t) = \sum_{n \in \mathbb{Z}} \text{rect}(t - 2n)$$

Example: Response of an RC Circuit to a Rectangular Pulse Train – Cont'd

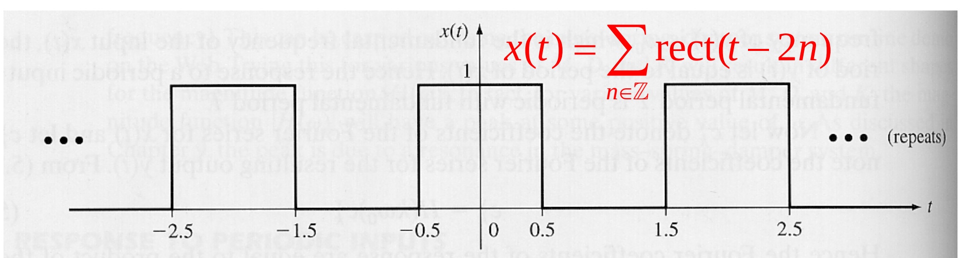


Figure 5.4 Periodic input signal in Example 5.4.

- We have found its Fourier series to be

$$x(t) = \sum_{k \in \mathbb{Z}} c_k^x e^{jk\pi t}, \quad t \in \mathbb{R}$$

with

$$c_k^x = \frac{1}{2} \text{sinc}\left(\frac{k}{2}\right)$$

Example: Response of an RC Circuit to a Rectangular Pulse Train – Cont'd

- Magnitude spectrum $|c_k^x|$ of input signal $x(t)$

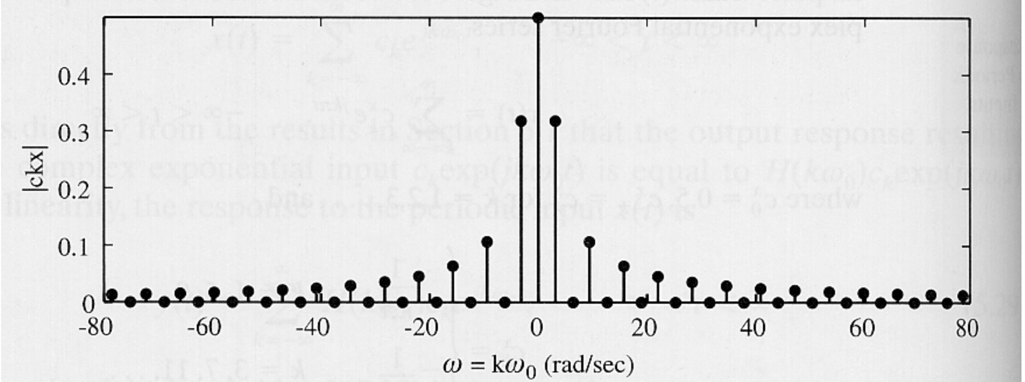


Figure 5.5 Amplitude spectrum of periodic input in Example 5.4.

Example: Response of an RC Circuit to a Rectangular Pulse Train – Cont'd

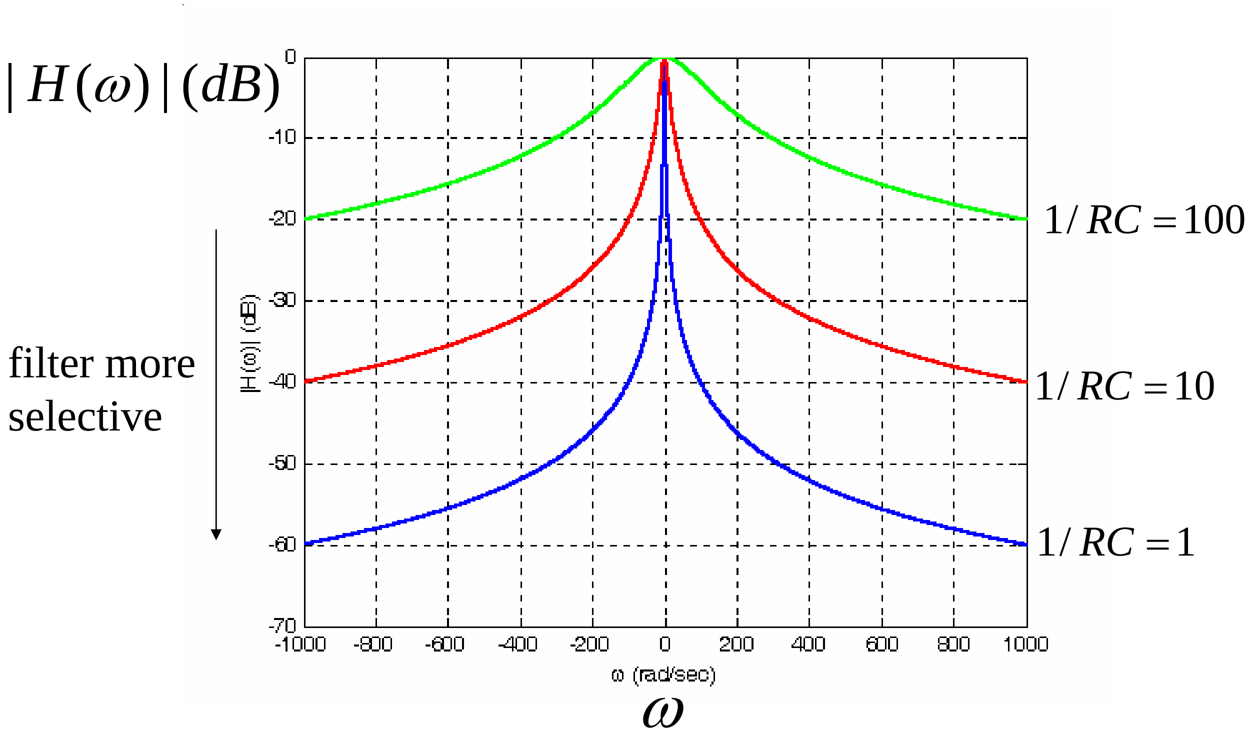
- The frequency response of the RC circuit was found to be

$$H(\omega) = \frac{1/RC}{j\omega + 1/RC}$$

- Thus, the Fourier series of the output signal is given by

$$y(t) = \sum_{k=-\infty}^{\infty} H(k\omega_0) c_k^x e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k^y e^{jk\omega_0 t}$$

Example: Response of an RC Circuit to a Rectangular Pulse Train – Cont'd



Example: Response of an RC Circuit to a Rectangular Pulse Train – Cont'd

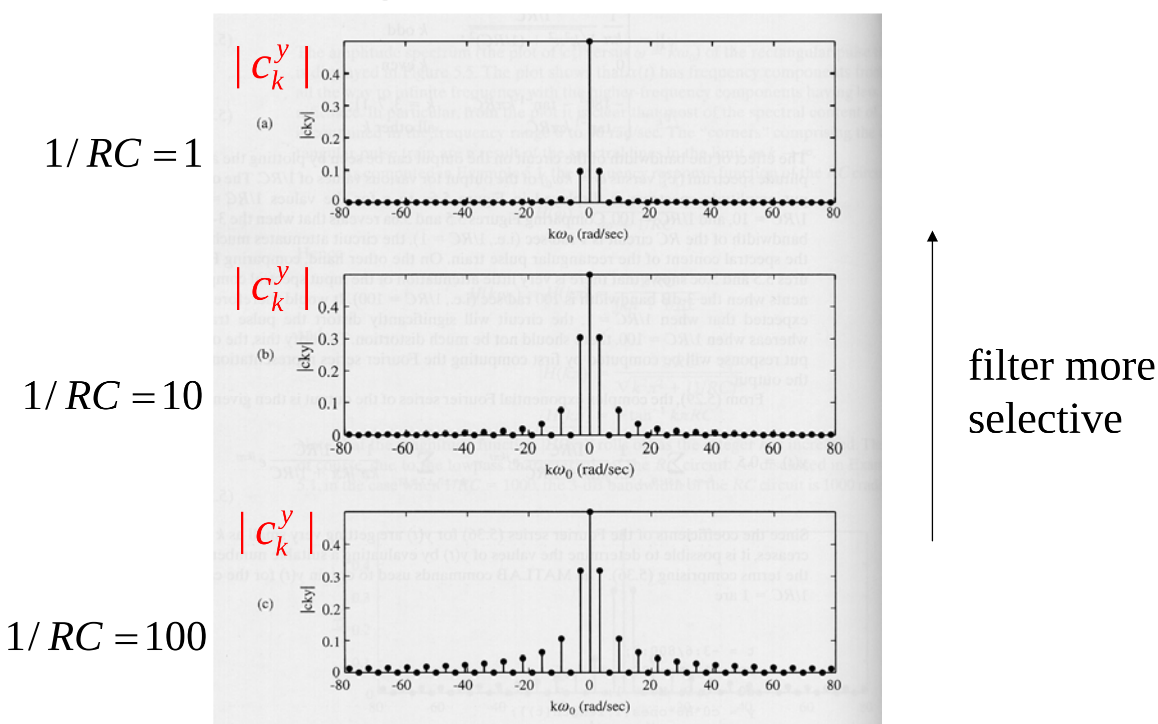


Figure 5.6 Amplitude spectrum of output when (a) $1/RC = 1$; (b) $1/RC = 10$; (c) $1/RC = 100$.

Example: Response of an RC Circuit to a Rectangular Pulse Train – Cont'd

$1/RC = 1$

$1/RC = 10$

$1/RC = 100$

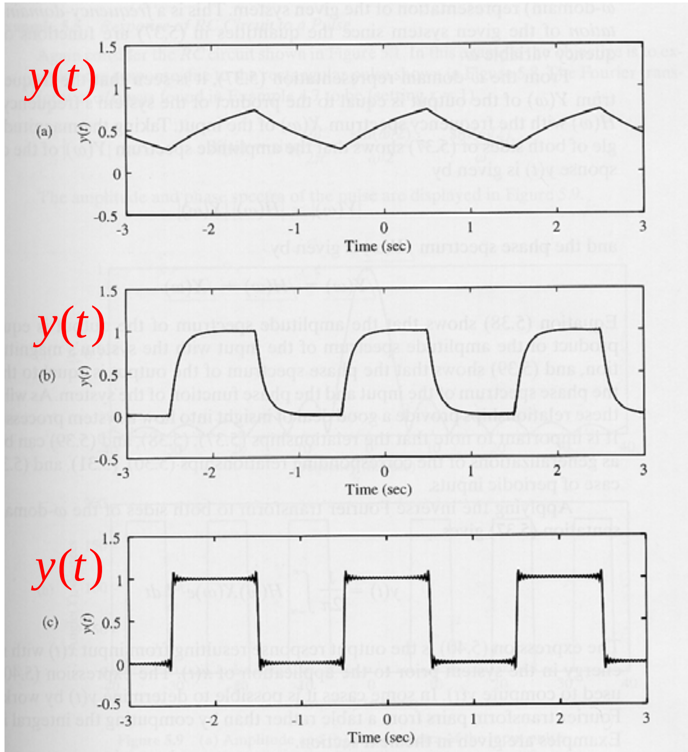
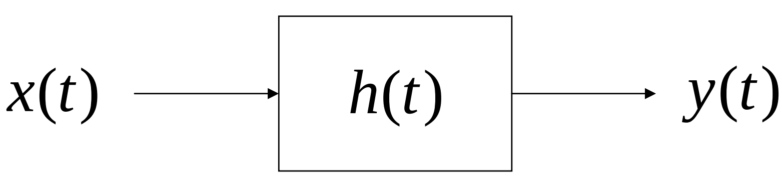


Figure 5.7 Plot of output when (a) $1/RC = 1$; (b) $1/RC = 10$; (c) $1/RC = 100$.

filter more selective

Response of a CT, LTI System to Aperiodic Inputs

- Consider the following CT, LTI system



- Its I/O relation is given by

$$y(t) = h(t) * x(t)$$

which, in the frequency domain, becomes

$$Y(\omega) = H(\omega)X(\omega)$$

Response of a CT, LTI System to Aperiodic Inputs – Cont'd

- From $Y(\omega) = H(\omega)X(\omega)$, the magnitude spectrum of the output signal $y(t)$ is given by

$$|Y(\omega)| = |H(\omega)||X(\omega)|$$

and its phase spectrum is given by

$$\arg Y(\omega) = \arg H(\omega) + \arg X(\omega)$$

Example: Response of an RC Circuit to a Rectangular Pulse

- Consider the RC circuit

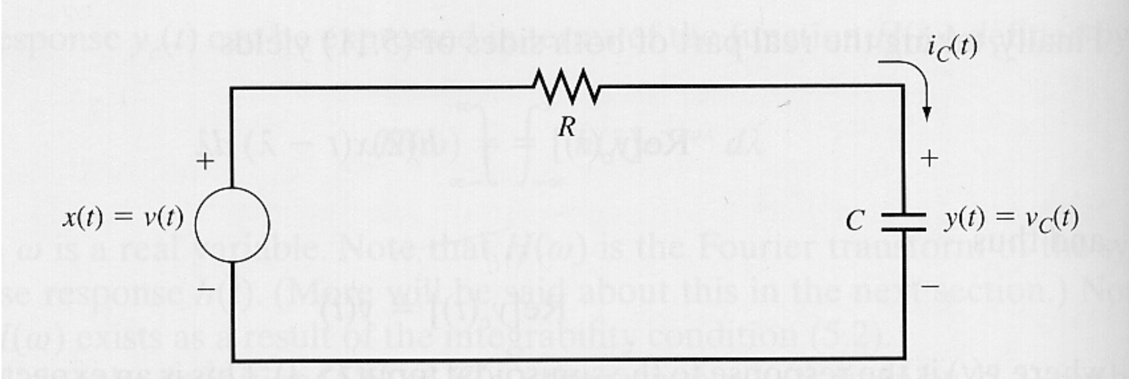


Figure 5.1 *RC* circuit in Example 5.2.

with input $x(t) = \text{rect}(t)$

Example: Response of an RC Circuit to a Rectangular Pulse – Cont'd

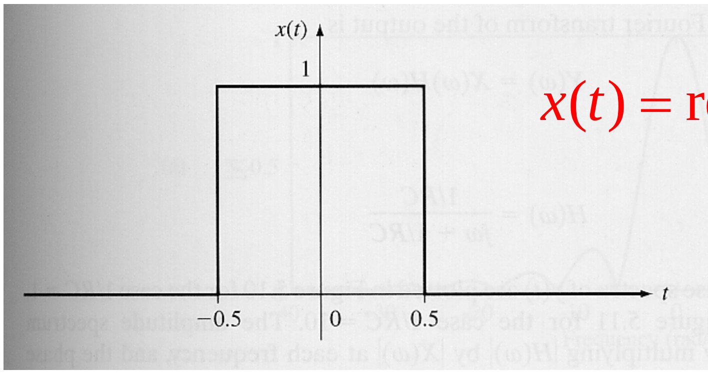


Figure 5.8 Input pulse in Example 5.5.

- The Fourier transform of $x(t)$ is

$$X(\omega) = \text{sinc}\left(\frac{\omega}{2\pi}\right)$$

Example: Response of an RC Circuit to a Rectangular Pulse – Cont'd

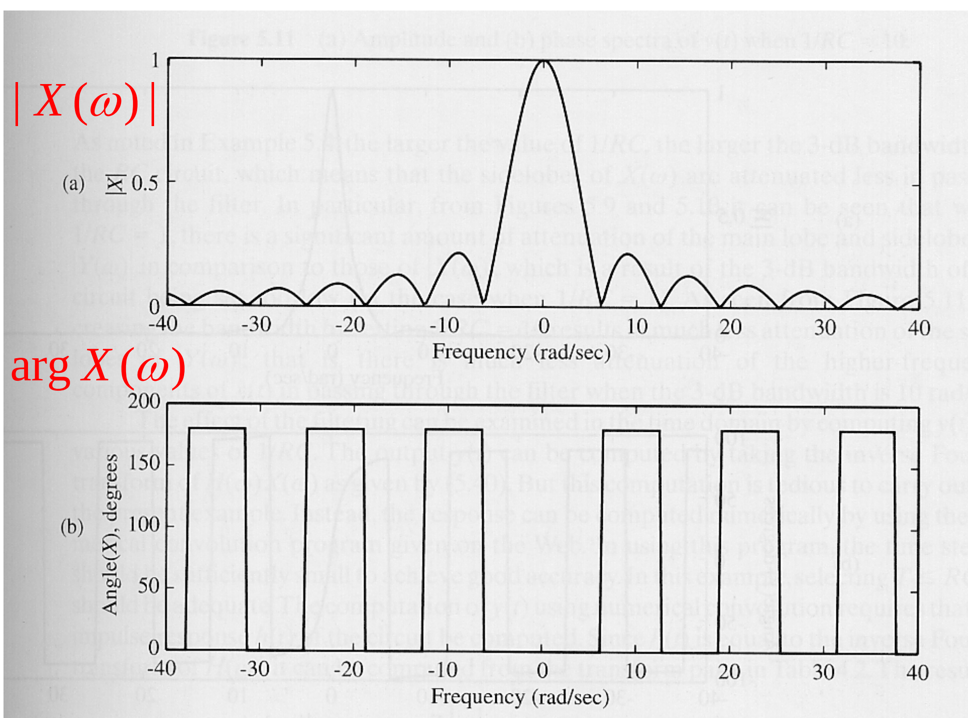


Figure 5.9 (a) Amplitude and (b) phase spectra of the input pulse.

Example: Response of an RC Circuit to a Rectangular Pulse – Cont'd

$1/RC = 1$

$|Y(\omega)|$

$\arg Y(\omega)$

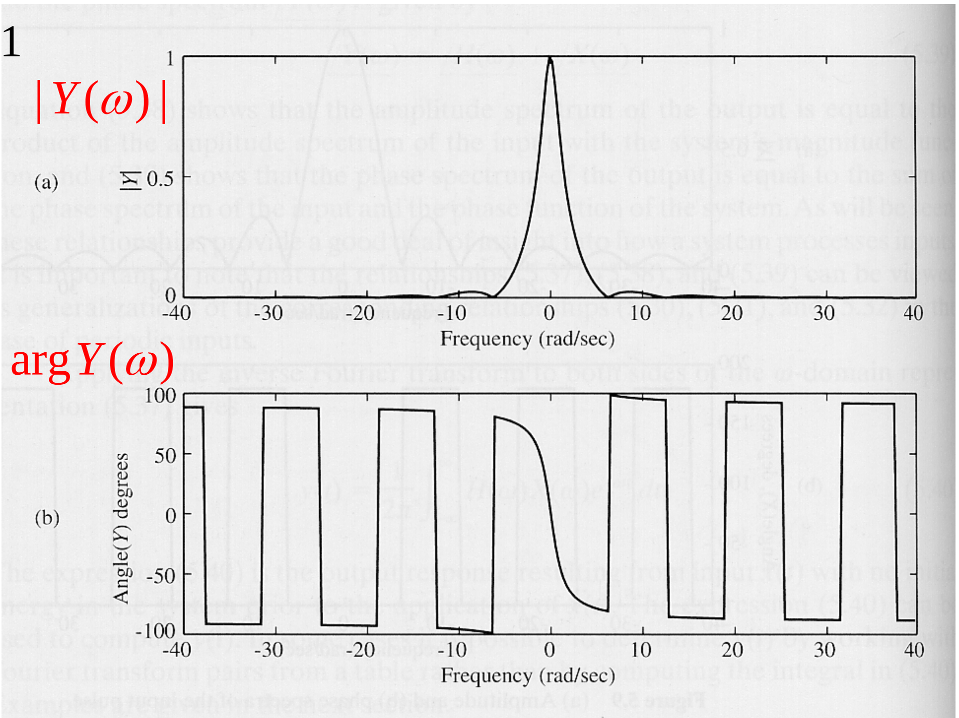


Figure 5.10 (a) Amplitude and (b) phase spectra of $y(t)$ when $1/RC = 1$.

Example: Response of an RC Circuit to a Rectangular Pulse – Cont'd

$1/RC = 10$

$|Y(\omega)|$

$\arg Y(\omega)$

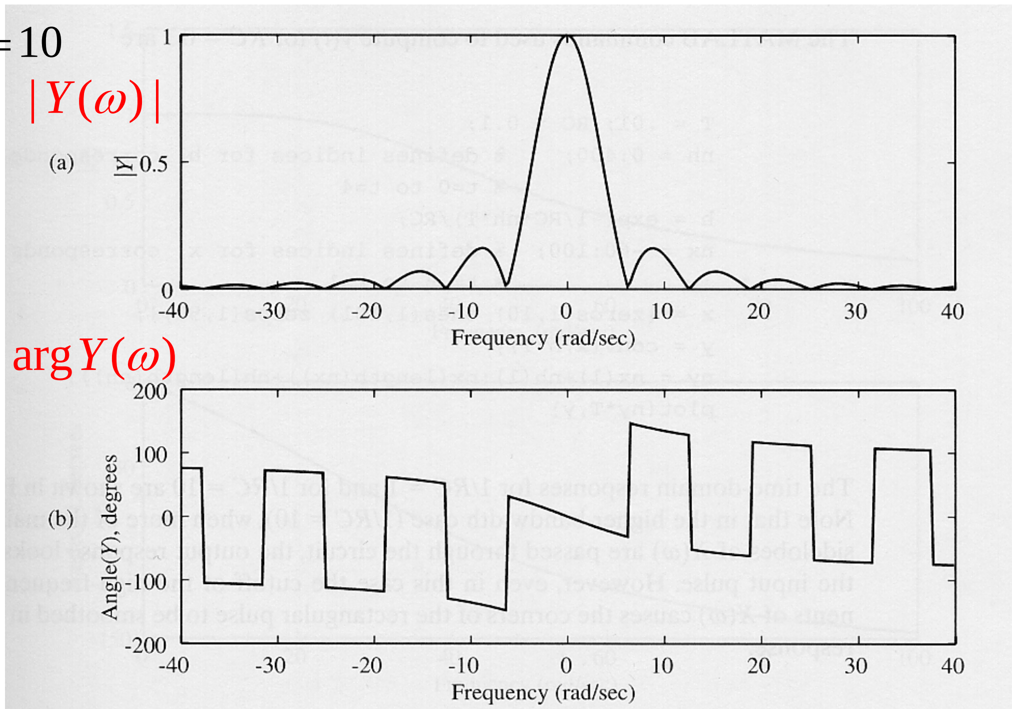


Figure 5.11 (a) Amplitude and (b) phase spectra of $y(t)$ when $1/RC = 10$.

Example: Response of an RC Circuit to a Rectangular Pulse – Cont'd

- The response of the system in the time domain can be found by computing the convolution

$$y(t) = h(t) * x(t)$$

where

$$h(t) = (1/RC)e^{-(1/RC)t}u(t)$$

$$x(t) = \mathrm{rect}(t)$$

Example: Response of an RC Circuit to a Rectangular Pulse – Cont'd

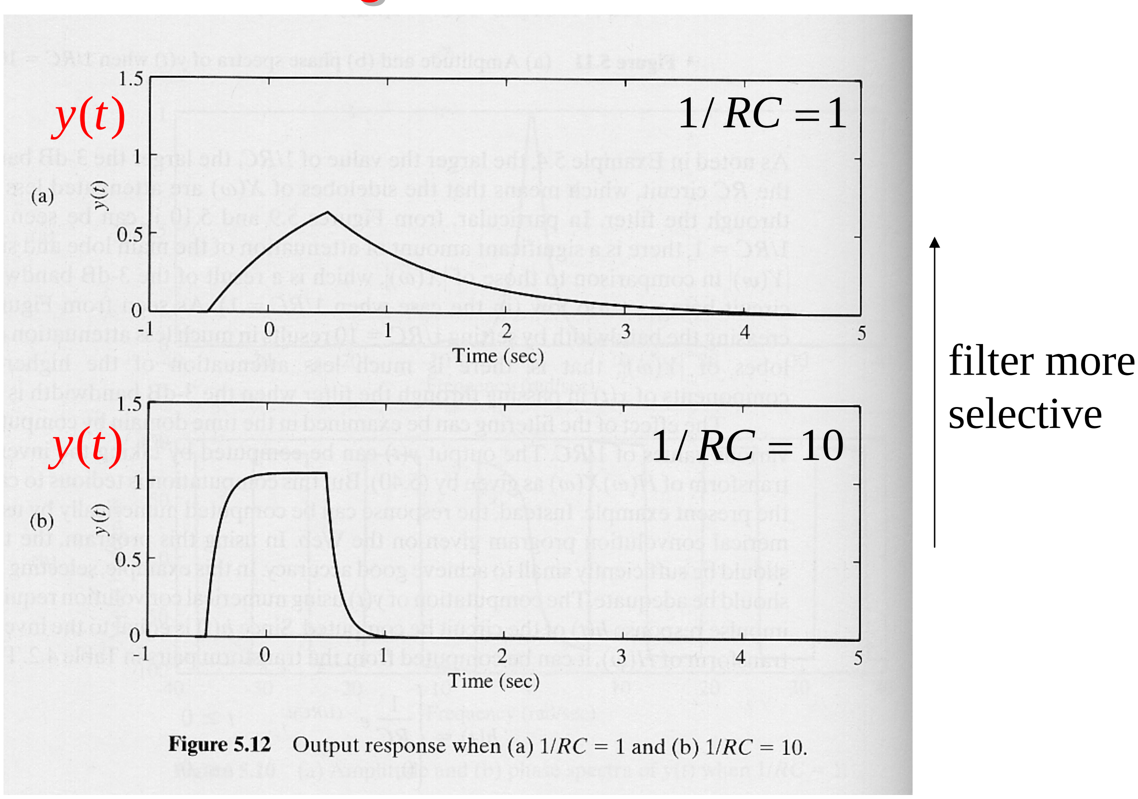


Figure 5.12 Output response when (a) $1/RC = 1$ and (b) $1/RC = 10$.

Example: Attenuation of High-Frequency Components

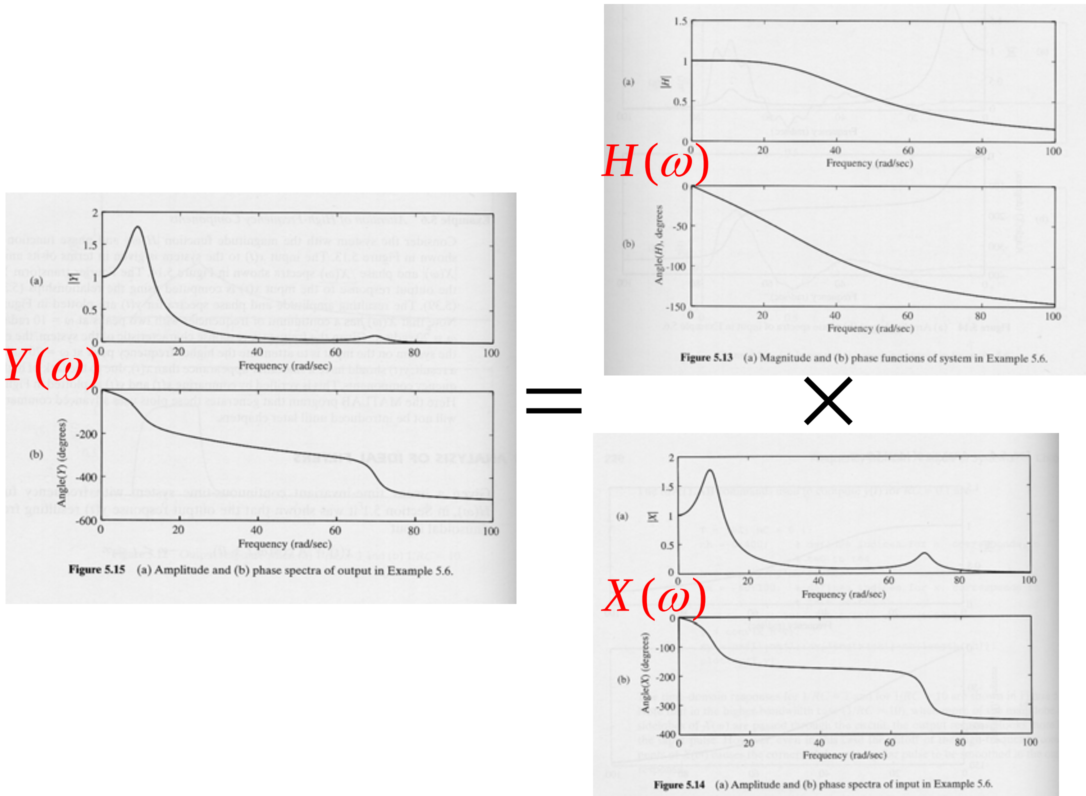


Figure 5.15 (a) Amplitude and (b) phase spectra of output in Example 5.6.

Figure 5.13 (a) Magnitude and (b) phase functions of system in Example 5.6.

Figure 5.14 (a) Amplitude and (b) phase spectra of input in Example 5.6.

Example: Attenuation of High-Frequency Components

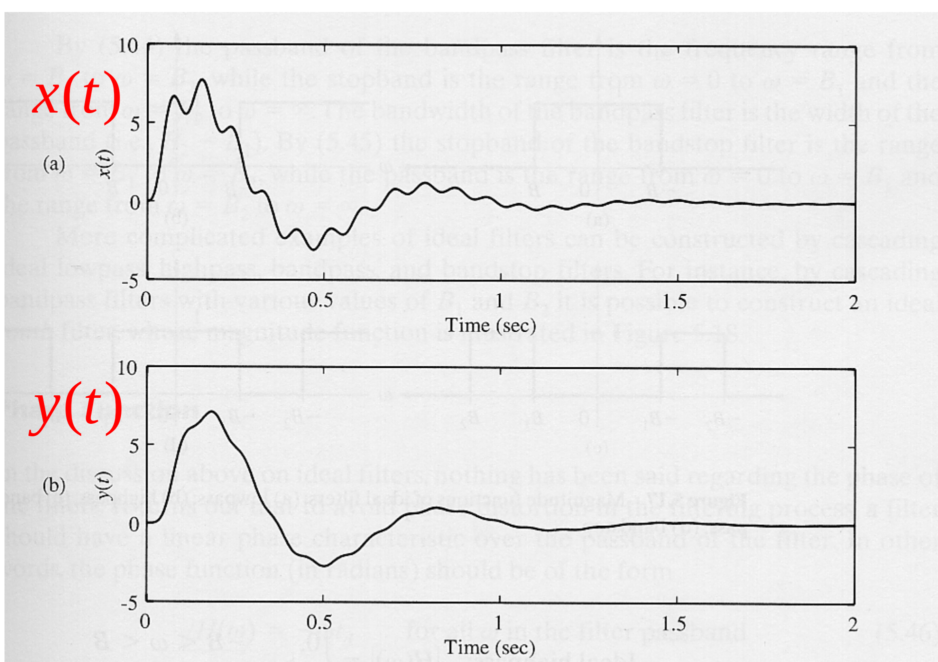


Figure 5.16 (a) Input and (b) resulting output in Example 5.6.

Filtering Signals

- The response of a CT, LTI system with frequency response $H(\omega)$ to a sinusoidal signal

$$x(t) = A\cos(\omega_0 t + \theta)$$

is

$$y(t) = A\,|H(\omega_0)|\cos\left(\omega_0 t + \theta + \arg H(\omega_0)\right)$$

- Filtering: if $|H(\omega_0)| = 0$ or $|H(\omega_0)| \approx 0$

then $y(t) = 0$ or $y(t) \approx 0, \ \forall t \in \mathbb{R}$

Four Basic Types of Filters

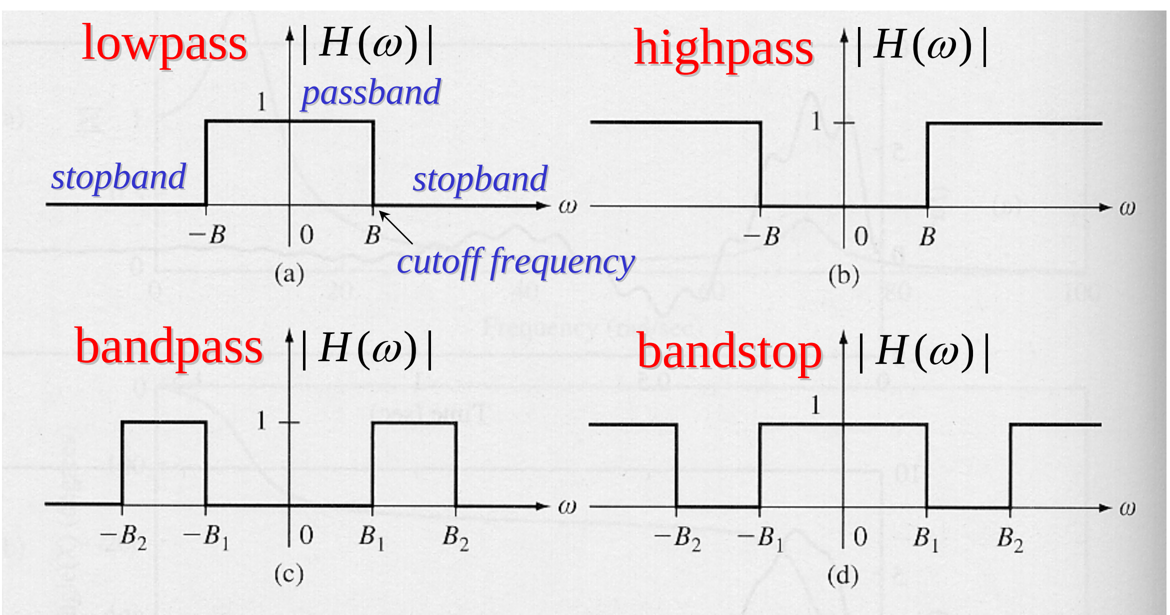


Figure 5.17 Magnitude functions of ideal filters: (a) lowpass; (b) highpass; (c) bandpass; (d) bandstop.

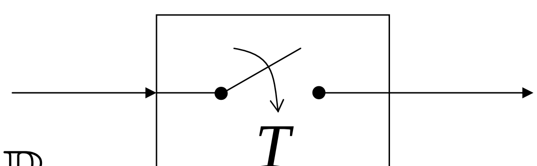
Phase Function

- Filters are usually designed based on specifications on the magnitude response $|H(\omega)|$
- The phase response $\arg H(\omega)$ has to be taken into account too in order to prevent signal distortion as the signal goes through the system
- If the filter has linear phase in its passband(s), then there is no distortion

Ideal Sampling

- Consider the ideal sampler:

$x(t)$, $t \in \mathbb{R}$



$$x[n] = x(t)\big|_{t=nT} = x(nT)$$

$n \in \mathbb{Z}$

- It is convenient to express the sampled signal $x(nT)$ as $x(t)p(t)$ where

$$p(t) = \sum_{n\in\mathbb{Z}} \delta(t - nT)$$

Ideal Sampling – Cont'd

- Thus, the sampled waveform $x(t)p(t)$ is

$$x(t)p(t) = \sum_{n\in\mathbb{Z}} x(t)\delta(t-nT) = \sum_{n\in\mathbb{Z}} x(nT)\delta(t-nT)$$

- $x(t)p(t)$ is an impulse train whose weights (areas) are the sample values $x(nT)$ of the original signal $x(t)$

Ideal Sampling – Cont'd

- Since $p(t)$ is periodic with period T, it can be represented by its Fourier series

$$p(t) = \sum_{k\in\mathbb{Z}} c_k e^{jk\omega_s t}, \quad \omega_s = \frac{2\pi}{T}$$

sampling frequency (rad/sec)

where $$c_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t)e^{-jk\omega_s t}dt, \quad k\in\mathbb{Z}$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_s t}dt = \frac{1}{T}$$

Ideal Sampling – Cont'd

- Therefore

$$p(t) = \sum_{k\in\mathbb{Z}} \frac{1}{T} e^{jk\omega_s t}$$

and

$$x_s(t) = x(t)\,p(t) = \sum_{k\in\mathbb{Z}} \frac{1}{T} x(t) e^{jk\omega_s t} = \frac{1}{T}\sum_{k\in\mathbb{Z}} x(t) e^{jk\omega_s t}$$

whose Fourier transform is

$$X_s(\omega) = \frac{1}{T}\sum_{k\in\mathbb{Z}} X(\omega - k\omega_s)$$

Ideal Sampling – Cont'd

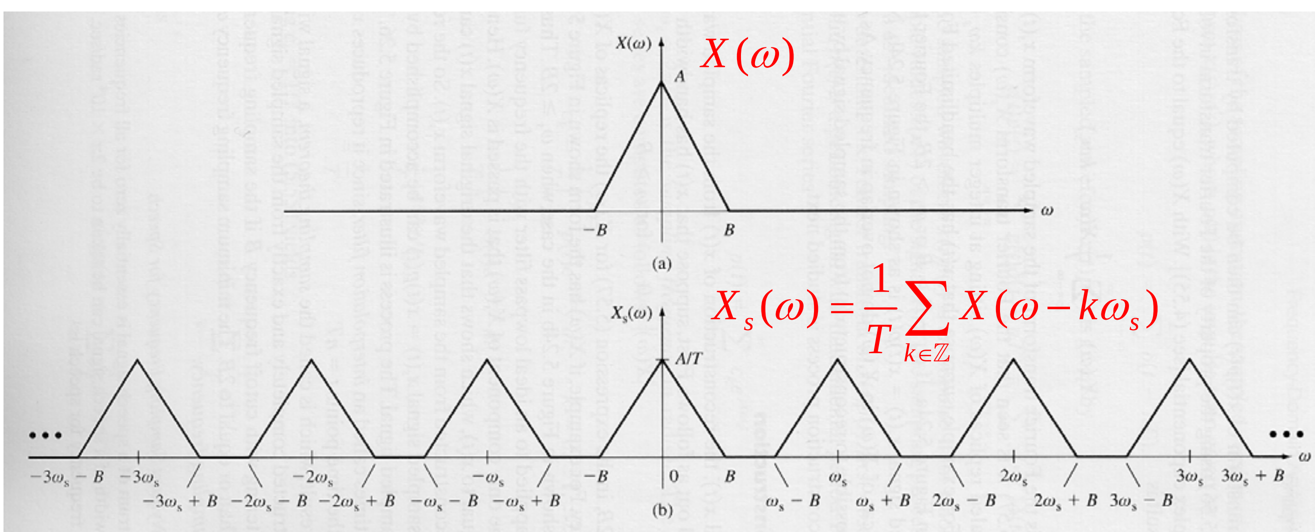


Figure 5.24 Fourier transform of (a) $x(t)$ and (b) $x_s(t) = x(t)p(t)$.

Signal Reconstruction

- Suppose that the signal $x(t)$ is bandlimited with bandwidth B, i.e., $|X(\omega)| = 0$, for $|\omega| > B$
- Then, if $\omega_s \geq 2B$, the replicas of $X(\omega)$ in

$$X_s(\omega) = \frac{1}{T}\sum_{k\in\mathbb{Z}} X(\omega - k\omega_s)$$

do not overlap and $X(\omega)$ can be recovered by applying an ideal lowpass filter to $X_s(\omega)$ (interpolation filter)

Interpolation Filter for Signal Reconstruction

$$H(\omega) = \begin{cases} T, & \omega \in [-B, B] \\ 0, & \omega \notin [-B, B] \end{cases}$$

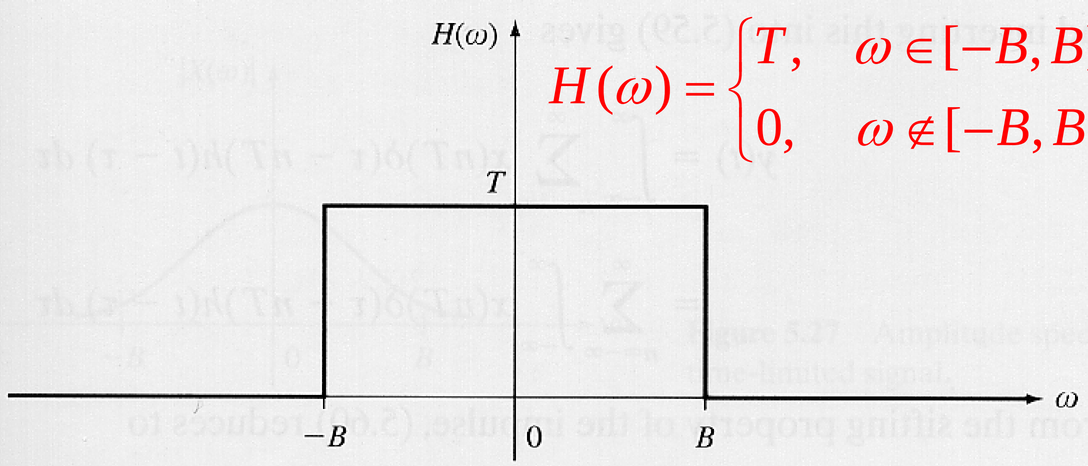


Figure 5.25 Frequency response function of ideal lowpass filter with bandwidth *B*.

Interpolation Formula

- The impulse response $h(t)$ of the interpolation filter is

$$h(t) = \frac{BT}{\pi}\operatorname{sinc}\left(\frac{\mathrm{B}}{\pi}t\right)$$

and the output $y(t)$ of the interpolation filter is given by

$$y(t) = h(t) * x_s(t)$$

Interpolation Formula – Cont'd

- But

$$x_s(t) = x(t)\,p(t) = \sum_{n\in\mathbb{Z}} x(nT)\delta(t-nT)$$

whence

$$y(t) = h(t) * x_s(t) = \sum_{n\in\mathbb{Z}} x(nT)h(t-nT) =$$

$$= \frac{BT}{\pi}\sum_{n\in\mathbb{Z}} x(nT)\operatorname{sinc}\left(\frac{B}{\pi}(t-nT)\right)$$

- Moreover, $y(t) = x(t)$

Shannon's Sampling Theorem

- A CT bandlimited signal $x(t)$ with frequencies no higher than B can be reconstructed from its samples $x[n] = x(nT)$ if the samples are taken at a rate

$$\omega_s = 2\pi / T \geq 2B$$

- The reconstruction of $x(t)$ from its samples $x[n] = x(nT)$ is provided by the interpolation formula

$$x(t) = \frac{BT}{\pi} \sum_{n \in \mathbb{Z}} x(nT) \operatorname{sinc}\left(\frac{B}{\pi}(t - nT) \right)$$

Nyquist Rate

- The minimum sampling rate $\omega_s = 2\pi / T = 2B$ is called the Nyquist rate

- Question: Why do CD's adopt a sampling rate of 44.1 *kHz*?
- Answer: Since the highest frequency perceived by humans is about 20 *kHz*, 44.1 *kHz* is slightly more than twice this upper bound

Aliasing

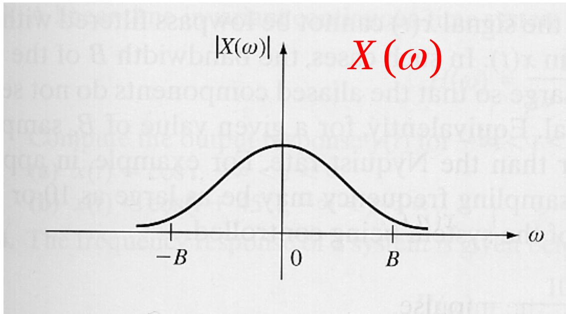


Figure 5.27 Amplitude spectrum of a time-limited signal.

$$X_s(\omega) = \frac{1}{T}\sum_{k\in\mathbb{Z}} X(\omega - k\omega_s)$$

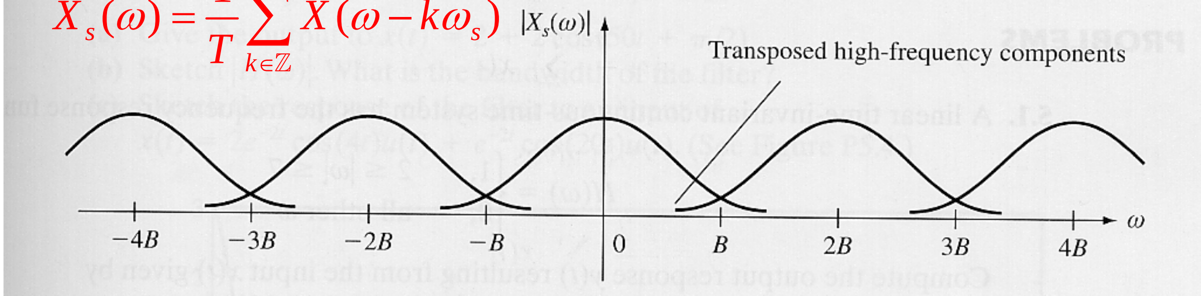


Figure 5.28 Amplitude spectrum of a sampled signal.

Aliasing –Cont'd

- Because of aliasing, it is not possible to reconstruct $x(t)$ exactly by lowpass filtering the sampled signal $x_s(t) = x(t)\,p(t)$
- Aliasing results in a distorted version of the original signal $x(t)$
- It can be eliminated (theoretically) by lowpass filtering $x(t)$ before sampling it so that $|X(\omega)| = 0$ for $|\omega| \geq B$