Chapter 3
Convolution Representation

CT Unit-Impulse Response

- Consider the CT SISO system:

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

- If the input signal is $x(t) = \delta(t)$ and the system has no energy at $t = 0^-$, the output $y(t) = h(t)$ is called the impulse response of the system

$$\delta(t) \longrightarrow \boxed{\text{System}} \longrightarrow h(t)$$

Exploiting Time-Invariance

- Let $x(t)$ be an arbitrary input signal with $x(t) = 0, \text{ for } t < 0$
- Using the sifting property of $\delta(t)$, we may write

$$x(t) = \int_{0^-}^{\infty} x(\tau)\delta(t-\tau)d\tau, \quad t \geq 0$$

- Exploiting time-invariance, it is

$$\delta(t-\tau) \longrightarrow \boxed{\text{System}} \longrightarrow h(t-\tau)$$

Exploiting Time-Invariance

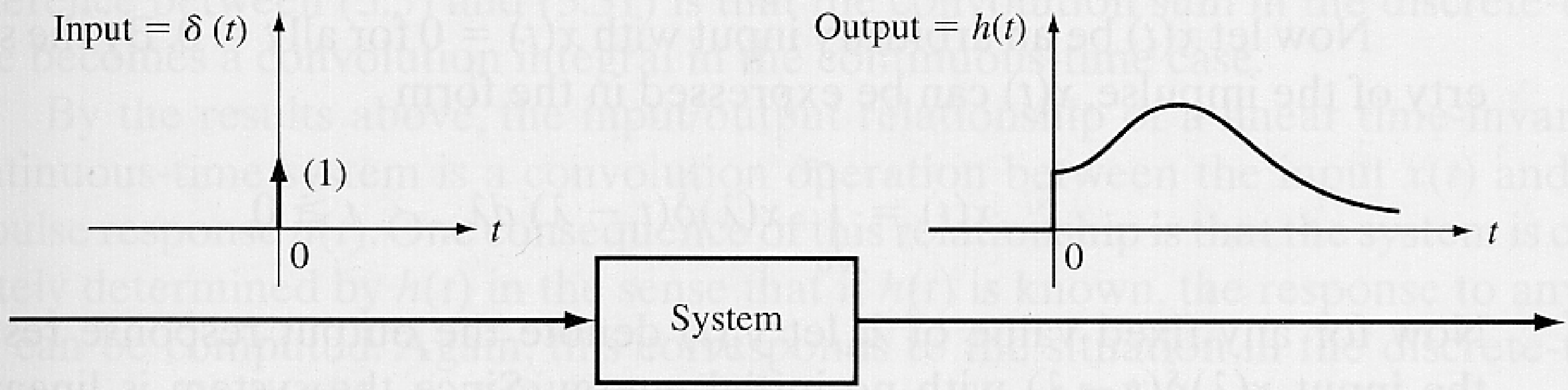


Figure 3.10 Generation of the impulse response.

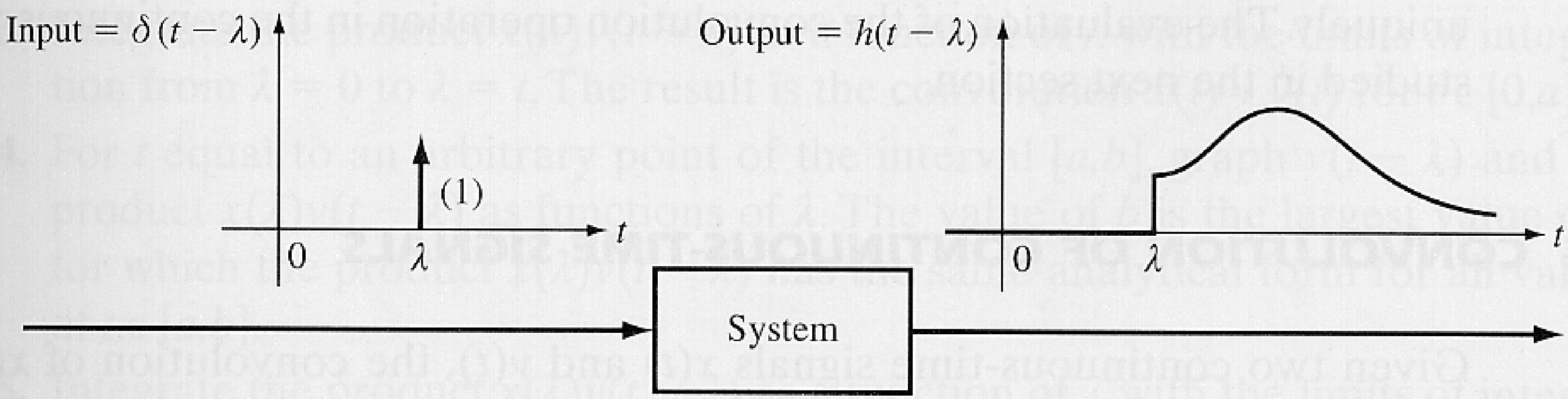


Figure 3.12 Time-shifted impulse response $h(t-\lambda)$.

Exploiting Linearity

- Exploiting linearity, it is

$$y(t) = \int_{0^-}^{\infty} x(\tau)h(t-\tau)d\tau, \quad t \geq 0$$

- If the integrand $x(\tau)h(t-\tau)$ does not contain an impulse located at $\tau = 0$, the lower limit of the integral can be taken to be 0,i.e.,

$$y(t) = \int_{0}^{\infty} x(\tau)h(t-\tau)d\tau, \quad t \geq 0$$

The Convolution Integral

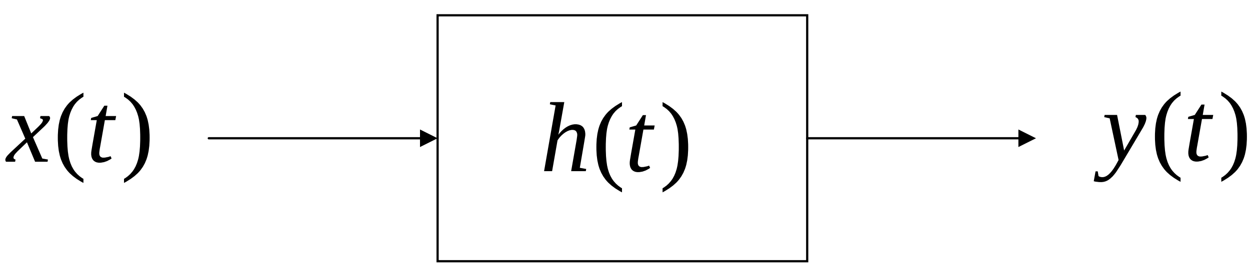
- This particular integration is called the convolution integral

$$y(t) = \underbrace{\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}_{x(t)*h(t)}, \quad t \geq 0$$

- Equation $y(t) = x(t) * h(t)$ is called the *convolution representation of the system*
- Remark: a CT LTI system is completely described by its impulse response $h(t)$

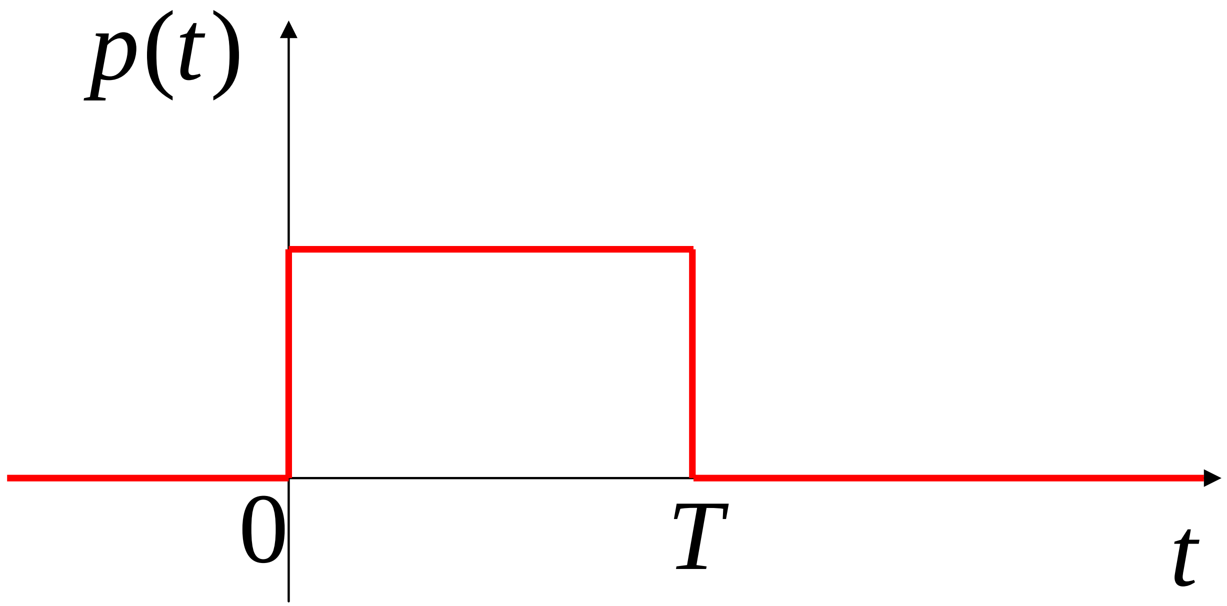
Block Diagram Representation of CT LTI Systems

- Since the impulse response $h(t)$ provides the complete description of a CT LTI system, we write



Example: Analytical Computation of the Convolution Integral

- Suppose that $x(t) = h(t) = p(t)$, where $p(t)$ is the rectangular pulse depicted in figure



Example – Cont'd

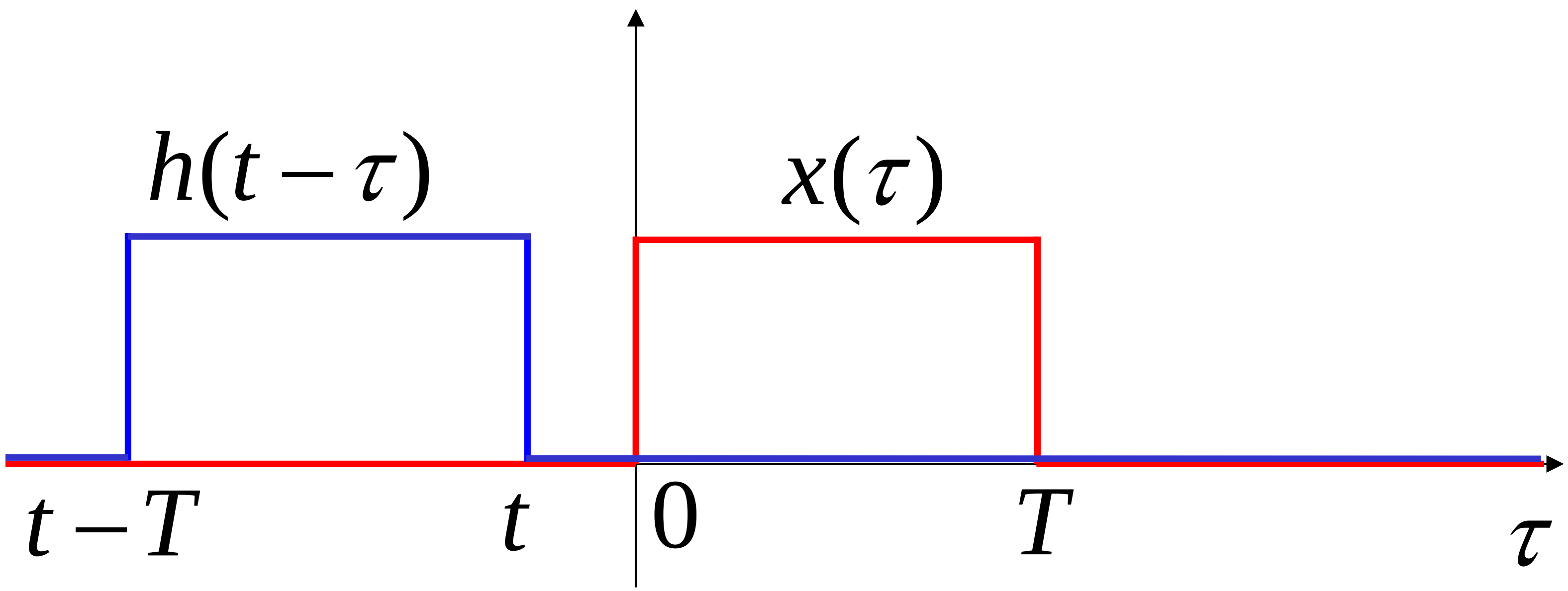
- In order to compute the convolution integral

$$y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau, \quad t \geq 0$$

we have to consider four cases:

Example – Cont'd

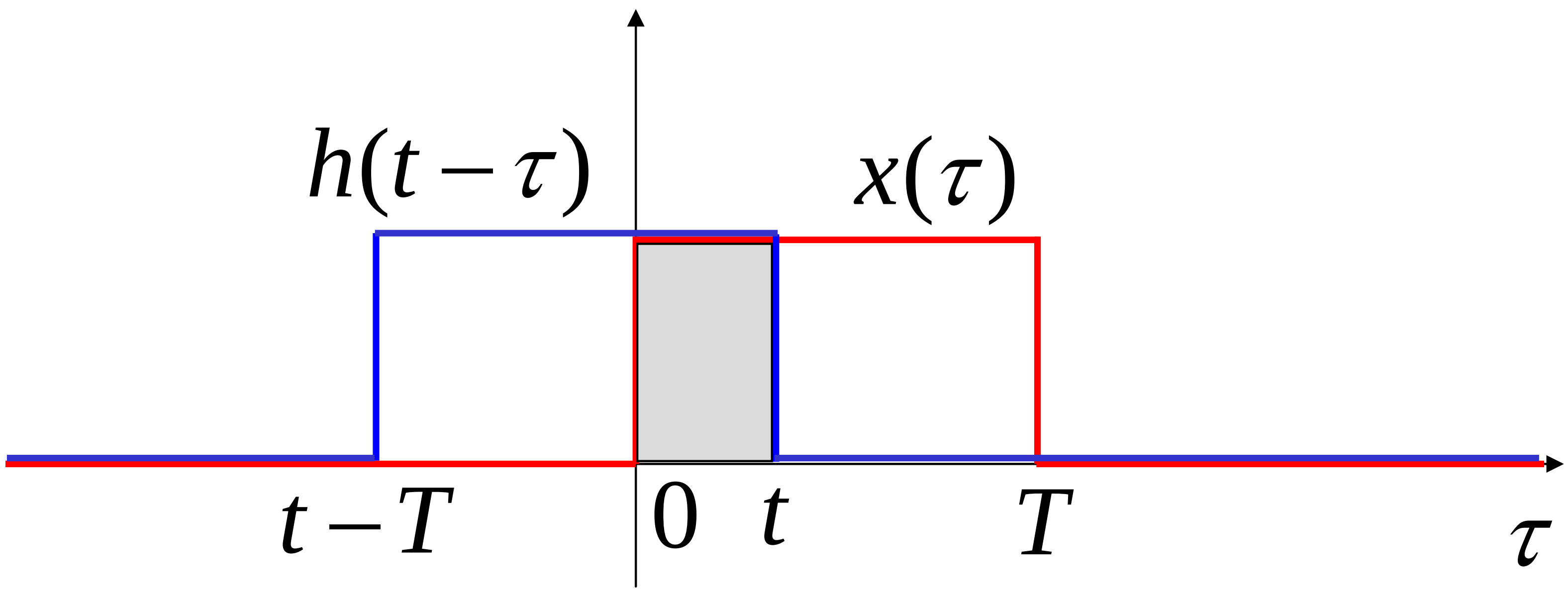
- Case 1: $t \leq 0$



$$y(t) = 0$$

Example – Cont'd

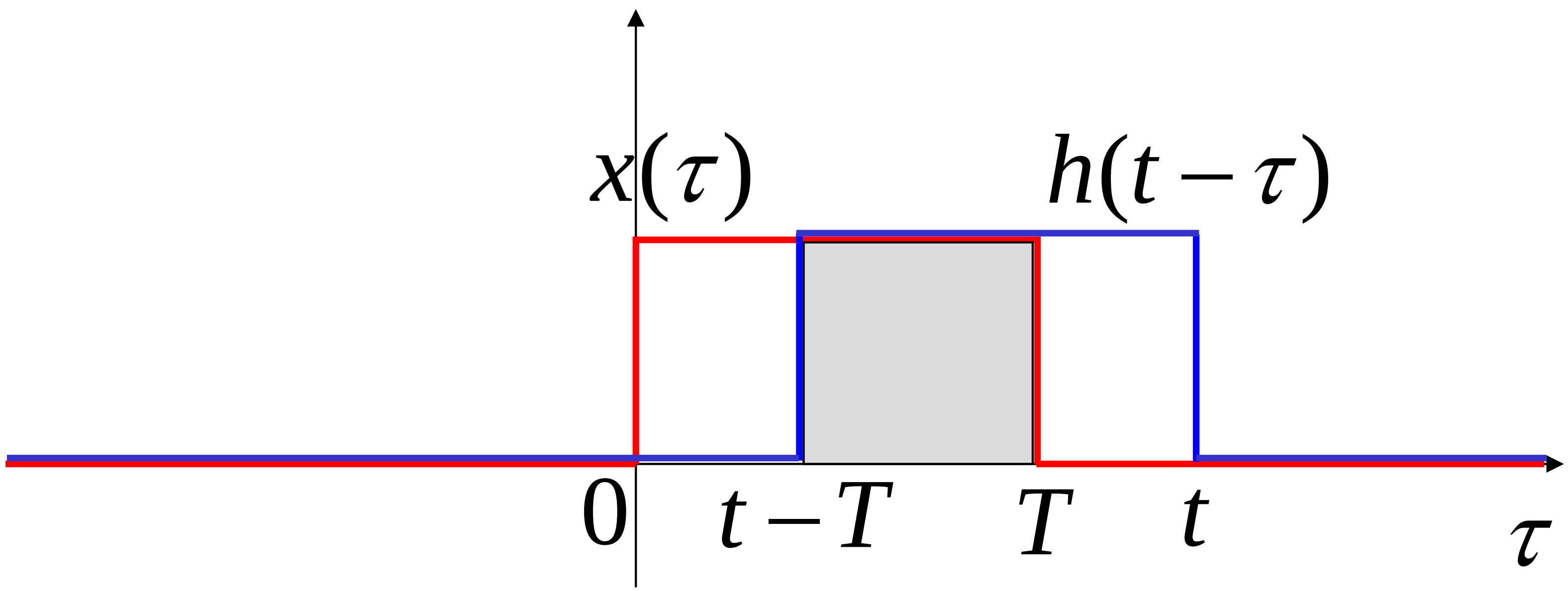
- Case 2: $0 \le t \le T$



$$y(t) = \int_0^t d\tau = t$$

Example – Cont'd

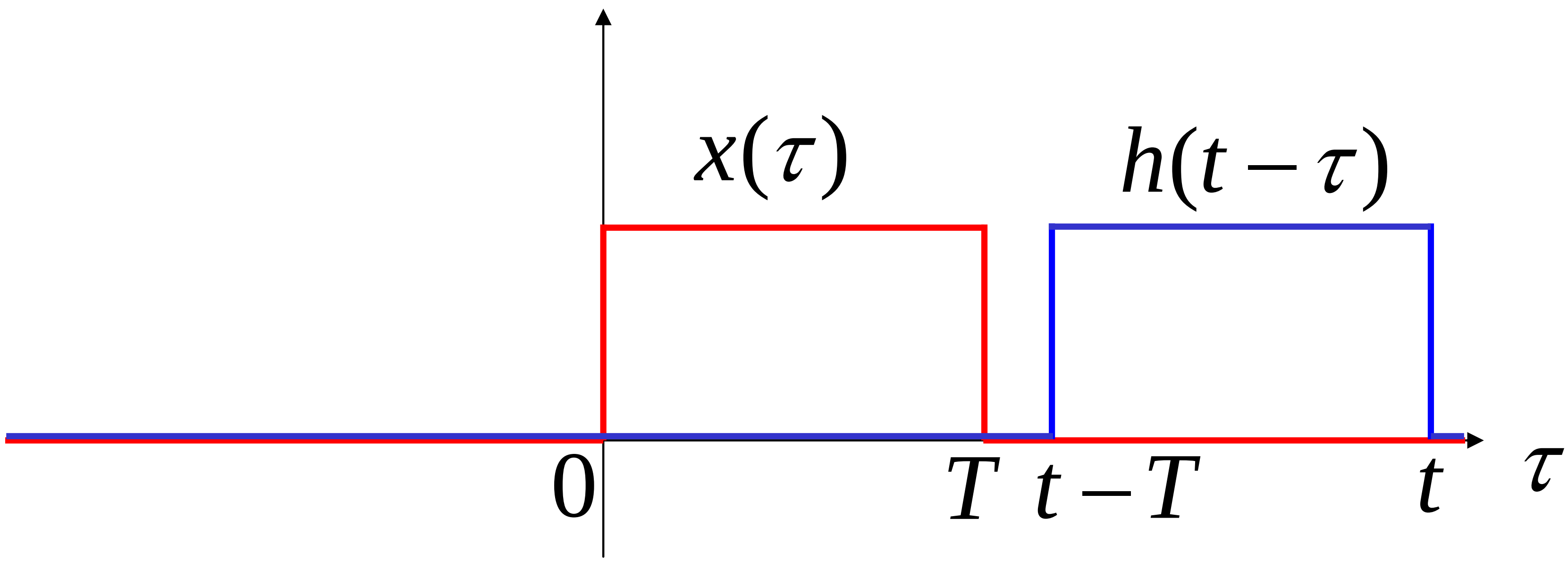
- Case 3: $0 \le t - T \le T \quad \rightarrow \quad T \le t \le 2T$



$$y(t) = \int_{t-T}^{T} d\tau = T - (t - T) = 2T - t$$

Example – Cont'd

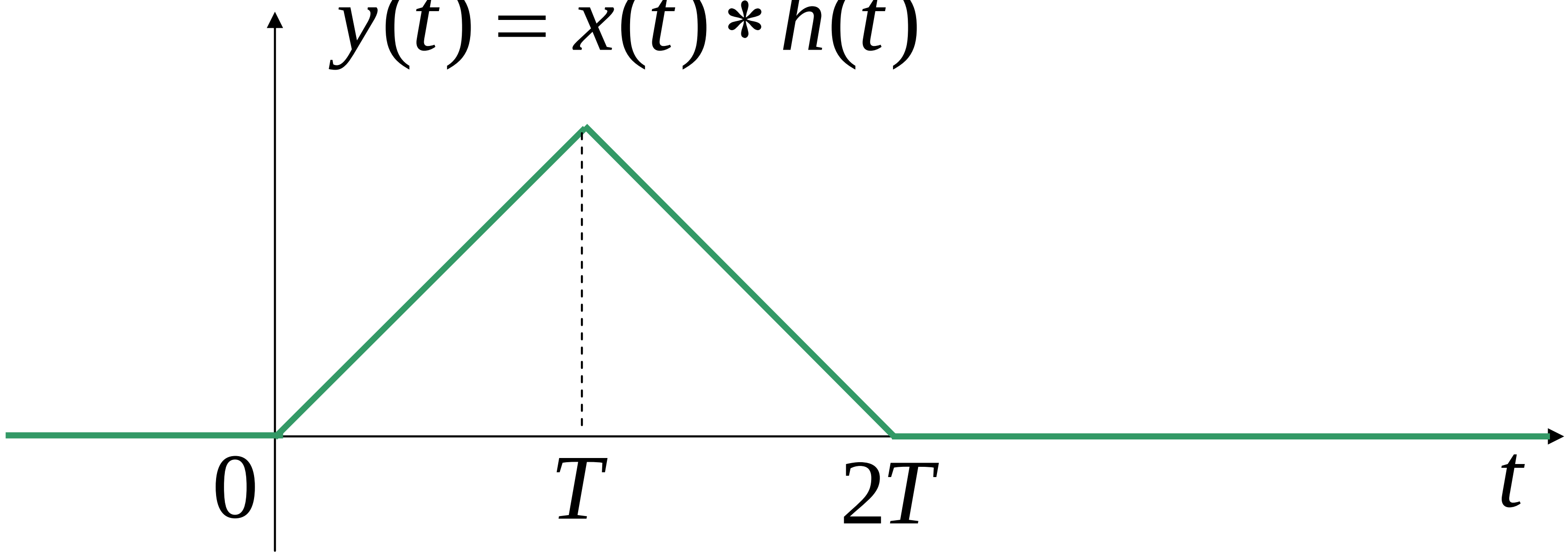
- Case 4: $T \leq t - T \quad \rightarrow \quad 2T \leq t$



$$y(t) = 0$$

Example – Cont'd

$$y(t) = x(t) * h(t)$$



Properties of the Convolution Integral

- Associativity

$$x(t) * (v(t) * w(t)) = (x(t) * v(t)) * w(t)$$

- Commutativity

$$x(t) * v(t) = v(t) * x(t)$$

- Distributivity w.r.t. addition

$$x(t) * (v(t) + w(t)) = x(t) * v(t) + x(t) * w(t)$$

Properties of the Convolution Integral - Cont'd

- Shift property: define

$$\begin{cases} x_q(t) = x(t-q) \\ v_q(t) = v(t-q) \\ w(t) = x(t) * v(t) \end{cases}$$

then

$$w(t-q) = x_q(t) * v(t) = x(t) * v_q(t)$$

- Convolution with the unit impulse

$$x(t) * \delta(t) = x(t)$$

- Convolution with the shifted unit impulse

$$x(t) * \delta_q(t) = x(t-q)$$

Properties of the Convolution Integral - Cont'd

- Derivative property: if the signal $x(t)$ is differentiable, then it is

$$\frac{d}{dt}\left[x(t)*v(t)\right]=\frac{dx(t)}{dt}*v(t)$$

- If both $x(t)$ and $v(t)$ are differentiable, then it is also

$$\frac{d^2}{dt^2}\left[x(t)*v(t)\right]=\frac{dx(t)}{dt}*\frac{dv(t)}{dt}$$

Properties of the Convolution Integral - Cont'd

- Integration property: define

$$\begin{cases} x^{(-1)}(t) \doteq \int\limits_{-\infty}^{t} x(\tau)d\tau \\ v^{(-1)}(t) \doteq \int\limits_{-\infty}^{t} v(\tau)d\tau \end{cases}$$

then

$$(x * v)^{(-1)}(t) = x^{(-1)}(t) * v(t) = x(t) * v^{(-1)}(t)$$

Representation of a CT LTI System in Terms of the Unit-Step Response

- Let $g(t)$ be the response of a system with impulse response $h(t)$ when $x(t) = u(t)$ with no initial energy at time $t = 0$, i.e.,

$$u(t) \longrightarrow \boxed{h(t)} \longrightarrow g(t)$$

- Therefore, it is

$$g(t) = h(t) * u(t)$$

Representation of a CT LTI System in Terms of the Unit-Step Response – Cont'd

- Differentiating both sides

$$\frac{dg(t)}{dt} = \frac{dh(t)}{dt} * u(t) = h(t) * \frac{du(t)}{dt}$$

- Recalling that

$$\frac{du(t)}{dt} = \delta(t) \quad \text{and} \quad h(t) = h(t) * \delta(t)$$

it is $$\frac{dg(t)}{dt} = h(t) \quad \text{or} \quad g(t) = \int_0^t h(\tau)d\tau$$