Chapter 1
Fundamental Concepts

Signals

- A signal is a pattern of variation of a physical quantity, often as a *function* of time (but also space, distance, position, etc).
- These quantities are usually the independent variables of the function defining the signal
- A signal encodes information, which is the variation itself

Signal Processing

- Signal processing is the discipline concerned with extracting, analyzing, and manipulating the information carried by signals
- The processing method depends on the type of signal and on the nature of the information carried by the signal

Characterization and Classification of Signals

- The type of signal depends on the nature of the independent variables and on the value of the function defining the signal
- For example, the independent variables can be continuous or discrete
- Likewise, the signal can be a continuous or discrete function of the independent variables

Characterization and Classification of Signals – Cont'd

- Moreover, the signal can be either a real-valued function or a complex-valued function
- A signal consisting of a single component is called a scalar or one-dimensional (1-D) signal

Examples: CT vs. DT Signals

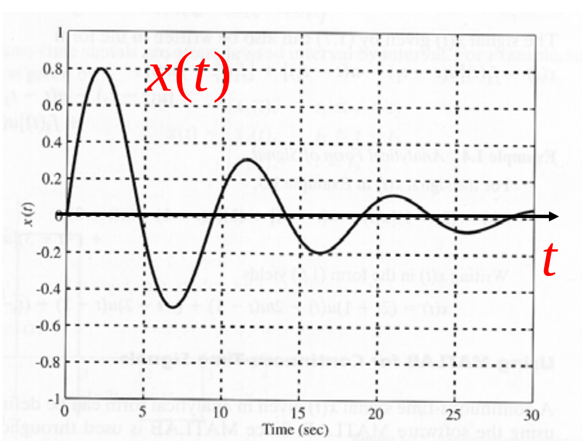


Figure 1.13 MATLAB plot of the signal $x(t) = e^{-0.1t} \sin \frac{2}{3}t$.

plot(t,x)

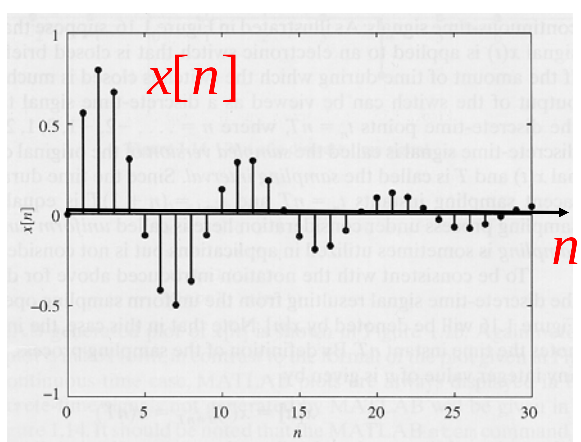


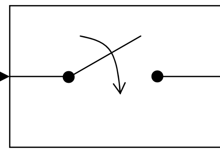
Figure 1.17 Sampled continuous-time signal.

stem(n,x)

Sampling

- Discrete-time signals are often obtained by sampling continuous-time signals

$x(t)$



$$x[n] = x(t)\big|_{t=nT} = x(nT)$$

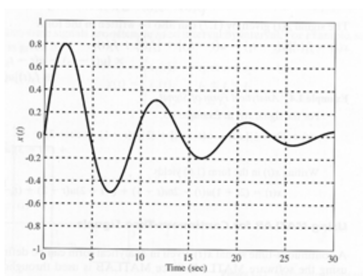


Figure 1.13 MATLAB plot of the signal $x(t) = e^{-0.1t} \sin \frac{2}{3} t$.

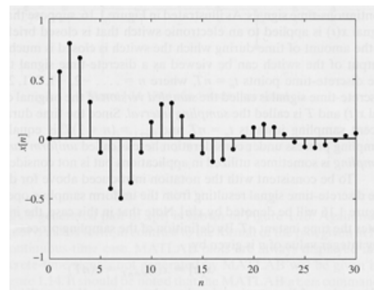
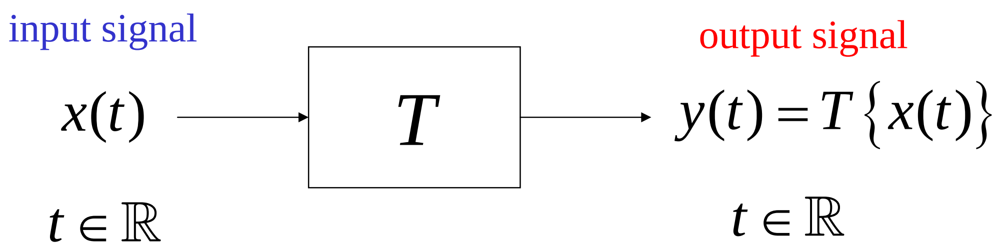


Figure 1.17 Sampled continuous-time signal.

Systems

- A system is any device that can process signals for analysis, synthesis, enhancement, format conversion, recording, transmission, etc.
- A system is usually mathematically defined by the equation(s) relating input to output signals (I/O characterization)
- A system may have single or multiple inputs and single or multiple outputs

Block Diagram Representation of Single-Input Single-Output (SISO) CT Systems



Types of input/output representations considered

- Differential equation
- Convolution model
- Transfer function representation (Fourier transform, Laplace transform)

Examples of 1-D, Real-Valued, CT Signals: Temporal Evolution of Currents and Voltages in Electrical Circuits

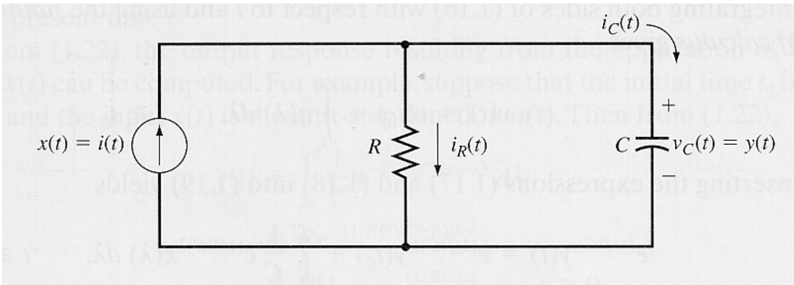


Figure 1.24 *RC* circuit.

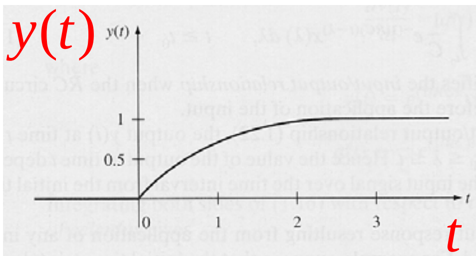


Figure 1.25 Step response of *RC* circuit when $R = C = 1$.

Examples of 1-D, Real-Valued, CT Signals: Temporal Evolution of Some Physical Quantities in Mechanical Systems

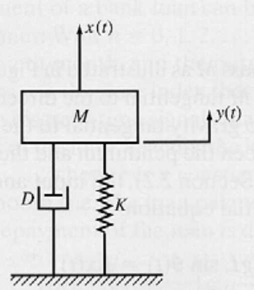
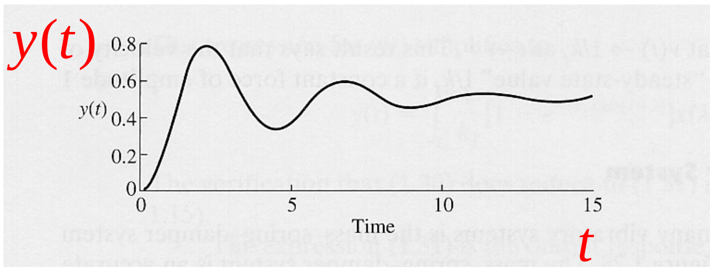


Figure 1.28 Schematic diagram of a mass–spring–damper system.



Continuous-Time (CT) Signals

- Unit-step function $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$
- Unit-ramp function $r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$

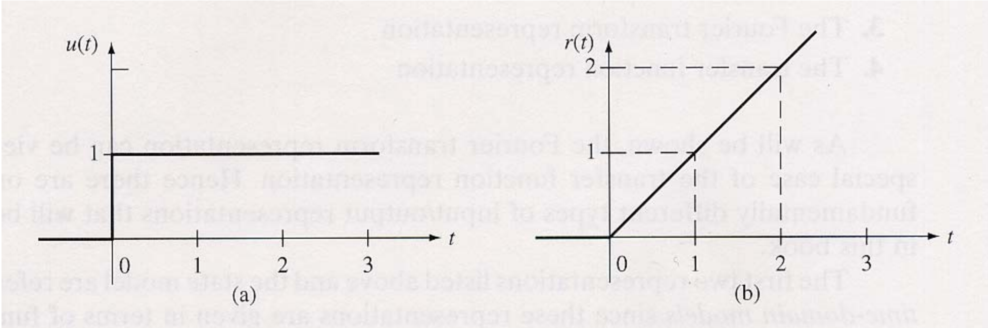


Figure 1.3 (a) Unit-step and (b) unit-ramp functions.

Unit-Ramp and Unit-Step Functions: Some Properties

$$x(t)u(t) = \begin{cases} x(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$r(t) = \int_{-\infty}^{t} u(\lambda) d\lambda$$

$$u(t) = \frac{dr(t)}{dt}$$ (with exception of $t = 0$)

The Rectangular Pulse Function

$$p_\tau(t) = u(t+\tau/2) - u(t-\tau/2)$$

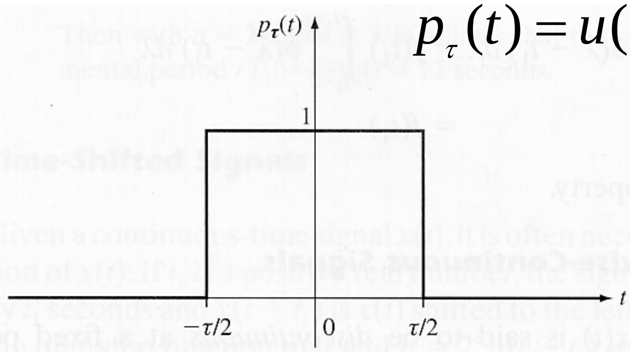


Figure 1.9 Rectangular pulse function.

The Unit Impulse

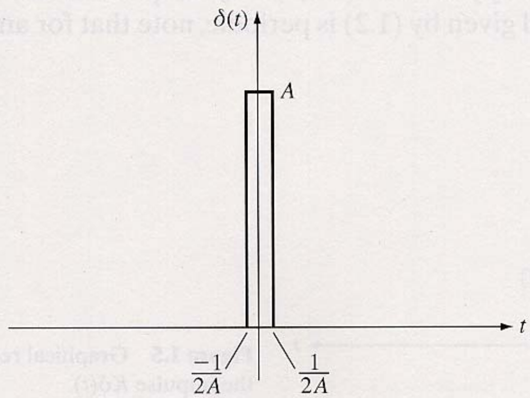
- A.k.a. the delta function or Dirac distribution
- It is defined by:

$$\delta(t) = 0, \quad t \neq 0$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(\lambda) d\lambda = 1, \quad \forall \varepsilon > 0$$

- The value $\delta(0)$ is not defined, in particular $\delta(0) \neq \infty$

The Unit Impulse: Graphical Interpretation



$$\delta(t) = \lim_{A \to \infty} p_{A(t)}$$

Figure 1.4 Pulse interpretation of $\delta(t)$.

A is a very large number

The Scaled Impulse $K\delta(t)$

- If $K \in \mathbb{R}$, $K\delta(t)$ is the impulse with area K, i.e.,

$$K\delta(t) = 0, \quad t \neq 0$$

$$\int_{-\varepsilon}^{\varepsilon} K\delta(\lambda)d\lambda = K, \quad \forall \varepsilon > 0$$

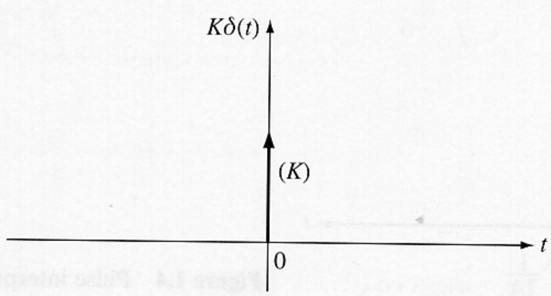


Figure 1.5 Graphical representation of the impulse $K\delta(t)$.

Properties of the Delta Function

1) $$u(t) = \int_{-\infty}^{t} \delta(\lambda)\,d\lambda$$

$$\forall t \text{ except } t = 0$$

2) $$\int_{t_0-\varepsilon}^{t_0+\varepsilon} x(t)\delta(t-t_0)\,dt = x(t_0) \quad \forall \varepsilon > 0$$

(sifting property)

Periodic Signals

- Definition: a signal $x(t)$ is said to be periodic with period T, if

$$x(t+T) = x(t) \quad \forall t \in \mathbb{R}$$

- Notice that $x(t)$ is also periodic with period qT where q is any positive integer
- T is called the fundamental period

Example: The Sinusoid

$$x(t) = A\cos(\omega t + \theta), \quad t \in \mathbb{R}$$

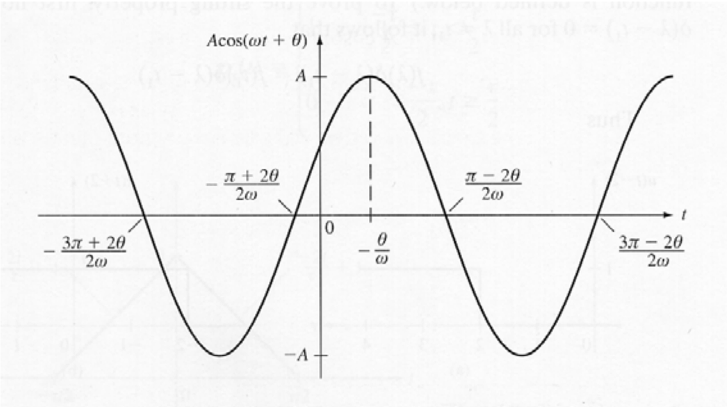


Figure 1.6 Sinusoid $A\cos(\omega t + \theta)$ with $-\pi/2 < \theta < 0$.

ω [rad / sec]

θ [rad]

$$f = \frac{\omega}{2\pi} \quad [1/\sec] = [Hz]$$

Time-Shifted Signals

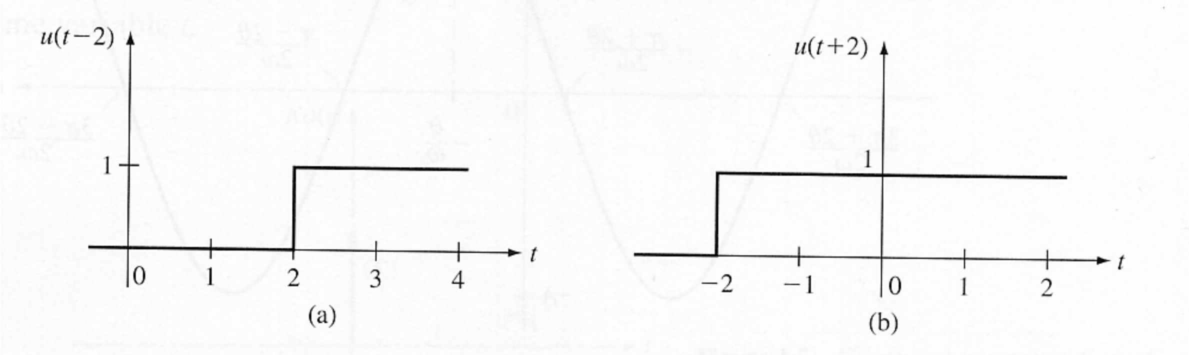
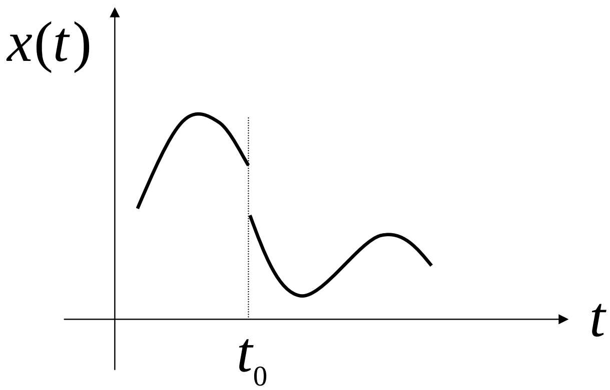


Figure 1.7 Two-second shifts of $u(t)$: (a) right shift; (b) left shift.

Points of Discontinuity

- A continuous-time signal $x(t)$ is said to be discontinuous at a point t_0 if $x(t_0^+) \neq x(t_0^-)$ where $t_0^+ = t_0 + \varepsilon$ and $t_0^- = t_0 - \varepsilon$, ε being a small positive number



Continuous Signals

- A signal $x(t)$ is continuous at the point t_0 if $x(t_0^+) = x(t_0^-)$

- If a signal $x(t)$ is continuous at all points t, $x(t)$ is said to be a continuous signal

Example of Continuous Signal: The Triangular Pulse Function

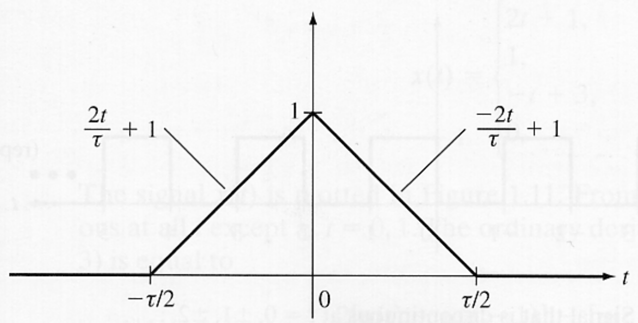


Figure 1.8 Triangular pulse function.

Piecewise-Continuous Signals

- A signal $x(t)$ is said to be piecewise continuous if it is continuous at all t except a finite or countably infinite collection of points $t_i, \, i = 1, 2, 3, \ldots$

Example of Piecewise-Continuous Signal: The Rectangular Pulse Function

$$p_\tau(t) = u(t+\tau/2) - u(t-\tau/2)$$

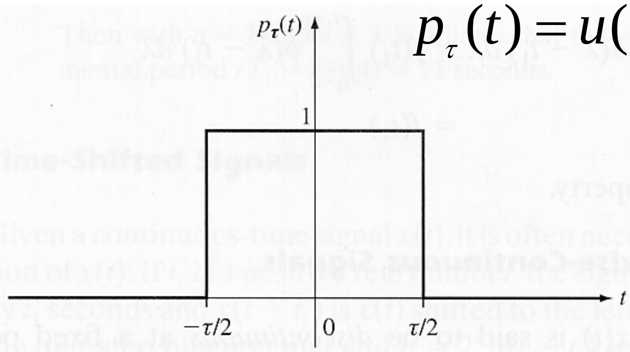


Figure 1.9 Rectangular pulse function.

Another Example of Piecewise-Continuous Signal: The Pulse Train Function

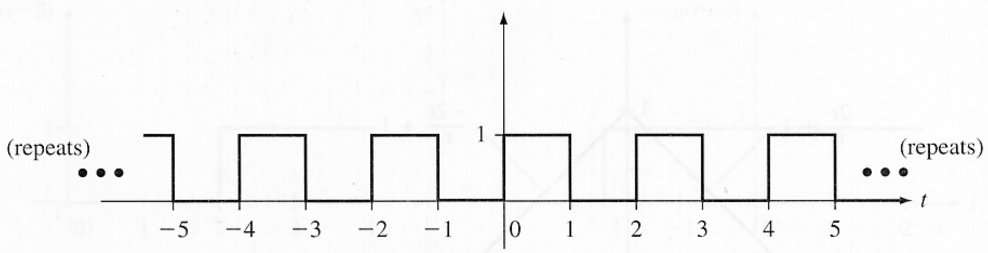


Figure 1.10 Signal that is discontinuous at $t = 0, \pm 1, \pm 2, \ldots$

Derivative of a Continuous-Time Signal

- A signal $x(t)$ is said to be differentiable at a point t_0 if the quantity

$$\frac{x(t_0+h)-x(t_0)}{h}$$

has limit as $h \to 0$ independent of whether h approaches 0 from above $(h>0)$ or from below $(h<0)$

- If the limit exists, $x(t)$ has a derivative at t_0

$$\left.\frac{dx(t)}{dt}\right|_{t=t_0} = \lim_{h\to 0}\frac{x(t_0+h)-x(t_0)}{h}$$

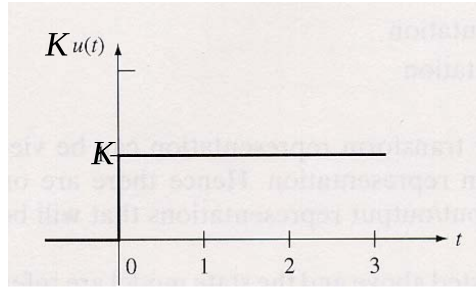
Generalized Derivative

- However, piecewise-continuous signals may have a derivative in a generalized sense
- Suppose that $x(t)$ is differentiable at all t except $t=t_0$
- The generalized derivative of $x(t)$ is defined to be

$$\underbrace{\frac{dx(t)}{dt}}_{\text{ordinary derivative of } x(t) \text{ at all } t \text{ except } t=t_0} + \left[x(t_0^+)-x(t_0^-)\right]\delta(t-t_0)$$

Example: Generalized Derivative of the Step Function

- Define $x(t) = Ku(t)$



- The ordinary derivative of $x(t)$ is 0 at all points except $t = 0$
- Therefore, the generalized derivative of $x(t)$ is

$$K\left[u(0^+) - u(0^-)\right]\delta(t-0) = K\delta(t)$$

Another Example of Generalized Derivative

- Consider the function defined as

$$x(t) = \begin{cases} 2t+1, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ -t+3, & 2 \le t \le 3 \\ 0, & \textit{all other } t \end{cases}$$

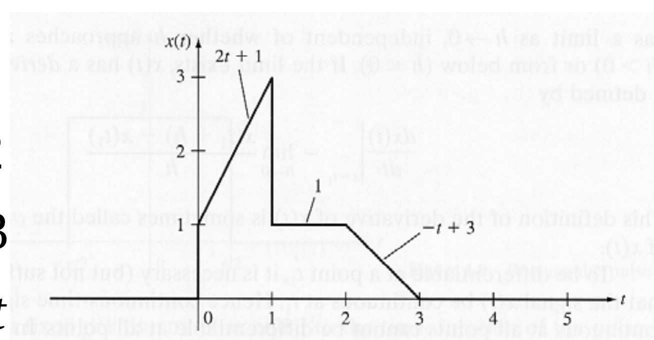


Figure 1.11 Signal in Example 1.3.

Another Example of Generalized Derivative: Cont'd

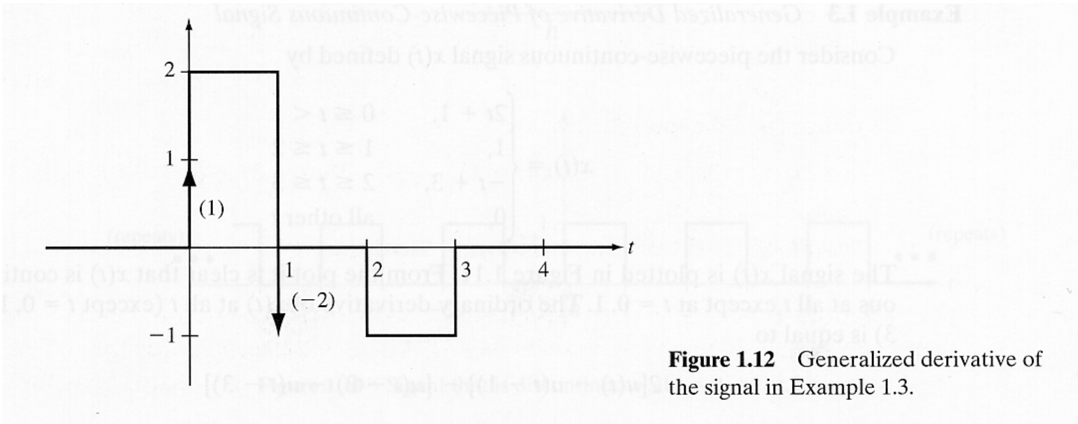


Figure 1.12 Generalized derivative of the signal in Example 1.3.

Example of CT System: An RC Circuit

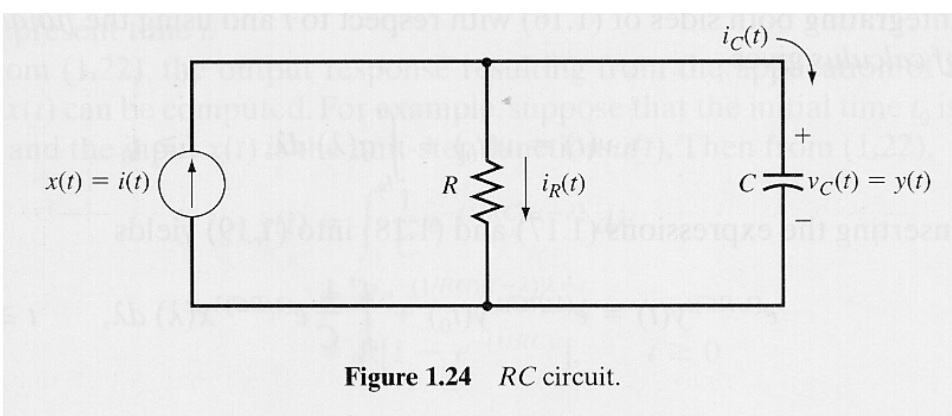


Figure 1.24 *RC* circuit.

Kirchhoff's current law: $i_C(t) + i_R(t) = i(t)$

RC Circuit: Cont'd

- The *v*-*i* law for the capacitor is

$$i_C(t) = C\frac{dv_C(t)}{dt} = C\frac{dy(t)}{dt}$$

- Whereas for the resistor it is

$$i_R(t) = \frac{1}{R}v_C(t) = \frac{1}{R}y(t)$$

RC Circuit: Cont'd

- Constant-coefficient linear differential equation describing the I/O relationship if the circuit

$$C\frac{dy(t)}{dt} + \frac{1}{R}y(t) = i(t) = x(t)$$

RC Circuit: Cont'd

- Step response when R=C=1

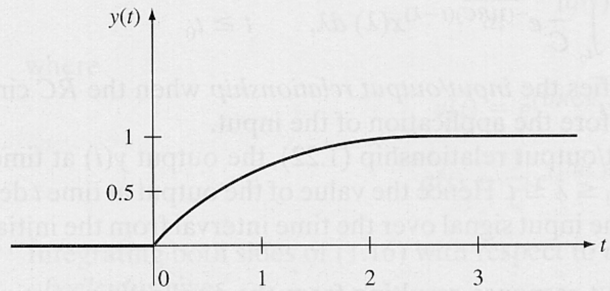


Figure 1.25 Step response of *RC* circuit when $R = C = 1$.

Basic System Properties: Causality

- A system is said to be causal if, for any time t_1, the output response at time t_1 resulting from input $x(t)$ does not depend on values of the input for $t > t_1$.
- A system is said to be noncausal if it is not causal

Example: The Ideal Predictor

$$y(t) = x(t+1)$$

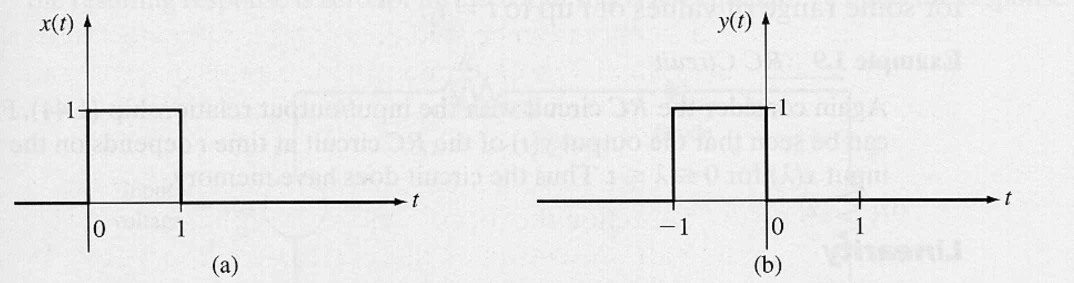


Figure 1.32 (a) Input and (b) output pulse of system in Example 1.5.

Example: The Ideal Delay

$$y(t) = x(t-1)$$

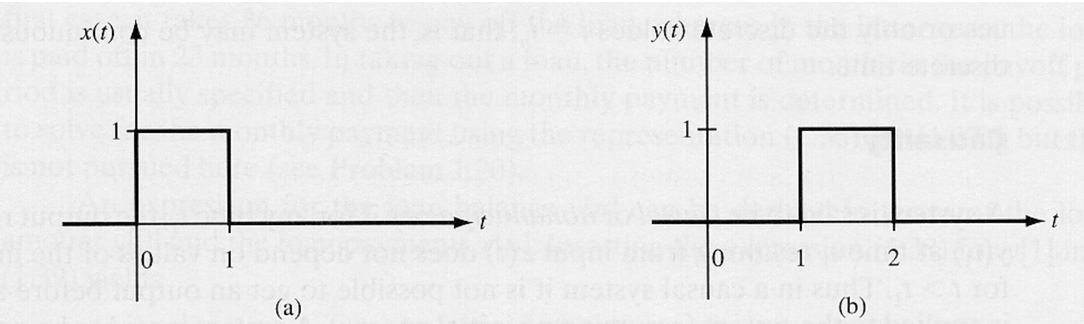


Figure 1.33 (a) Input and (b) output pulse of system in Example 1.6.

Memoryless Systems and Systems with Memory

- A causal system is memoryless or static if, for any time t_1, the value of the output at time t_1 depends only on the value of the input at time t_1

- A causal system that is not memoryless is said to have memory. A system has memory if the output at time t_1 depends in general on the past values of the input $x(t)$ for some range of values of t up to $t = t_1$

Examples

- Ideal Amplifier/Attenuator

$$y(t) = Kx(t)$$

- RC Circuit

$$y(t) = \frac{1}{C}\int_0^t e^{-(1/RC)(t-\tau)} x(\tau)\,d\tau, \quad t \geq 0$$

Basic System Properties: Additive Systems

- A system is said to be additive if, for any two inputs $x_1(t)$ and $x_2(t)$, the response to the sum of inputs $x_1(t) + x_2(t)$ is equal to the sum of the responses to the inputs (assuming no initial energy before the application of the inputs)

$$x_1(t)+x_2(t) \longrightarrow \boxed{\text{system}} \longrightarrow y_1(t)+y_2(t)$$

Basic System Properties: Homogeneous Systems

- A system is said to be homogeneous if, for any input $x(t)$ and any scalar a, the response to the input $ax(t)$ is equal to a times the response to $x(t)$, assuming no energy before the application of the input

$$ax(t) \longrightarrow \boxed{\text{system}} \longrightarrow ay(t)$$

Basic System Properties: Linearity

- A system is said to be linear if it is both additive and homogeneous

$$ax_1(t)+bx_2(t) \longrightarrow \boxed{\text{system}} \longrightarrow ay_1(t)+by_2(t)$$

- A system that is not linear is said to be nonlinear

Example of Nonlinear System: Circuit with a Diode

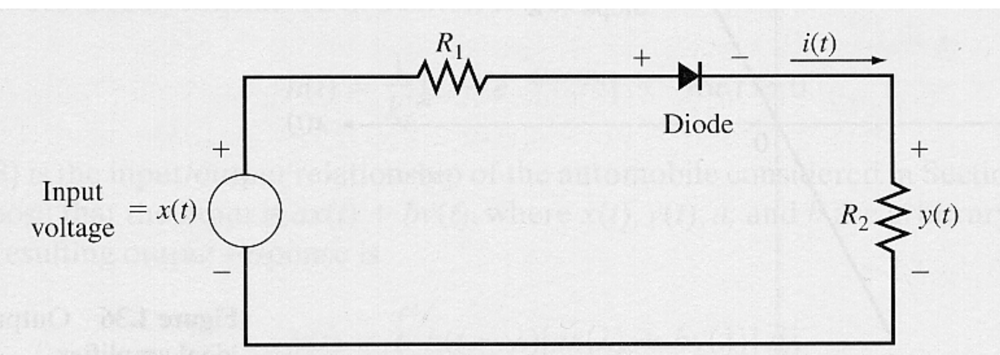


Figure 1.34 Resistive circuit with an ideal diode.

$$y(t)=\begin{cases} \dfrac{R_2}{R_1+R_2}x(t), & when\ x(t)\geq 0 \\ 0, & when\ x(t)\leq 0 \end{cases}$$

Example of Nonlinear System: Square-Law Device

$$y(t) = x^2(t)$$

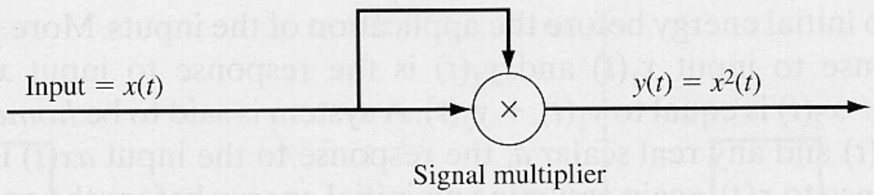


Figure 1.35 Realization of $y(t) = x^2(t)$.

Example of Linear System: The Ideal Amplifier

$$y(t) = Kx(t)$$

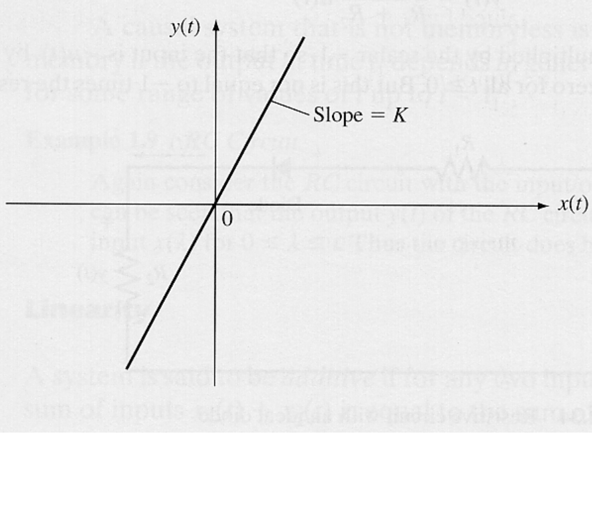


Figure 1.36 Output versus input in an ideal amplifier.

Example of Nonlinear System: A Real Amplifier

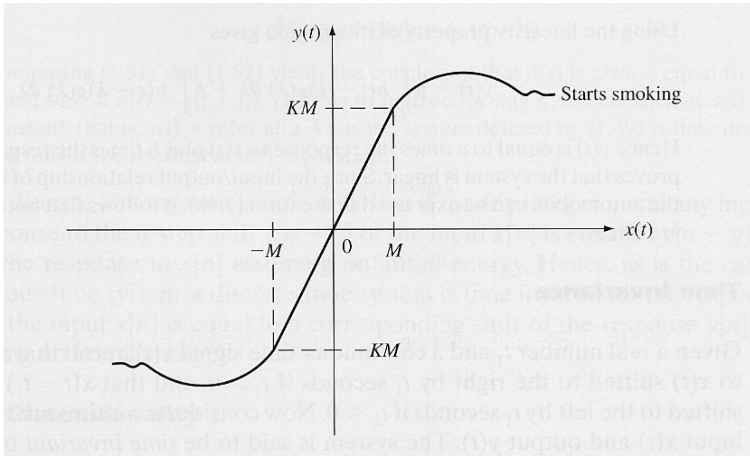


Figure 1.37 Output versus input in a nonideal amplifier.

Basic System Properties: Time Invariance

- A system is said to be time invariant if, for any input $x(t)$ and any time t_1, the response to the shifted input $x(t-t_1)$ is equal to $y(t-t_1)$ where $y(t)$ is the response to $x(t)$ with zero initial energy

$$x(t-t_1) \longrightarrow \boxed{\text{system}} \longrightarrow y(t-t_1)$$

- A system that is not time invariant is said to be time varying or time variant

Examples of Time Varying Systems

- Amplifier with Time-Varying Gain

$$y(t) = tx(t)$$

- First-Order System

$$\dot{y}(t) + a(t)y(t) = bx(t)$$

Basic System Properties: CT Linear Finite-Dimensional Systems

- If the N-th derivative of a CT system can be written in the form

$$y^{(N)}(t) = -\sum_{i=0}^{N-1} a_i(t) y^{(i)}(t) + \sum_{i=0}^{M} b_i(t) x^{(i)}(t)$$

then the system is both linear and finite dimensional

- To be time-invariant

$$a_i(t) = a_i \quad and \quad b_i(t) = b_i \quad \forall\ i\ and\ t \in \mathbb{R}$$