Fractal Coding Performance for First Order Gauss-Markov Models

B E Wohlberg and G de Jager
Digital Image Processing Laboratory,
Electrical Engineering Department,
University of Cape Town,
Private Bag, Rondebosch 7700, SOUTH AFRICA
brendt@dip1.ee.uct.ac.za

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Abstract

Fixed block size fractal coding is evaluated for first-order Gauss-Markov models, and the effects of varying the correlation are presented. Performance for this class of statistical models is found to be poor compared with traditional techniques such as transform coding.

1 Introduction

In fractal coding [1] of signals (usually images), each signal is represented by the coefficients of a contractive affine transform on itself. The signal is tiled by non-overlapping *range* blocks, and larger, possibly overlapping *domain* blocks. The global affine mapping consists of scaling and offset coefficients mapping one of the domain blocks (after averaging to the range block width) to each range block. These coefficients are identified by use of the Collage Theorem [2], which bounds the *real error* on decoding in terms of the *collage error*. While coding performance evaluations utilising test images have been promising [2], the performance of this coding method has not previously been evaluated for a statistical signal source, leaving fractal coding at a disadvantage in comparison with traditional methods such as transform coding.

2 Signal Model

A Gaussian first-order Markov (or AR(1)) process $X(n)$ is generated [3, ch. 2] by

$$X(n) = Z(n) + \rho X(n-1)$$

where ρ is the correlation and $Z(n)$ are independently distributed Gaussian values with variance σ_z^2. The autocorrelation function of this process is $R_{xx}(k) = \sigma_x^2 \rho^{|k|}$, with $\sigma_z^2 = (1-\rho^2)\sigma_x^2$. Transform coding may be shown to be close to optimal for these models [3].

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The performance of fixed block size fractal coding schemes for this model were investigated by calculating the distortion for each member of an ensemble of 1000 signals (restricted to 1-dimension to reduce computational requirements) randomly generated according to the model. Results are quoted for signal size n, range block size r, domain block size $2r$ with the domain pool consisting of all domain blocks an increment of 2 samples apart, and contractivity bound s_{max} (for all scaling s, $|s| < s_{max}$). All distortions are Mean Square Error (MSE), with $\sigma_z^2 = 1$ constant as ρ varies.

3 Real and Collage Errors

Figure 1 illustrates the difference between the collage and real error for signals of 4096 samples. In these and all other cases the real error was significantly greater than the collage error, with a decrease in the difference with increasing ρ. Since the collage error may be calculated more rapidly than the real error, it is used as a lower bound for the real error in many subsequent comparisons.

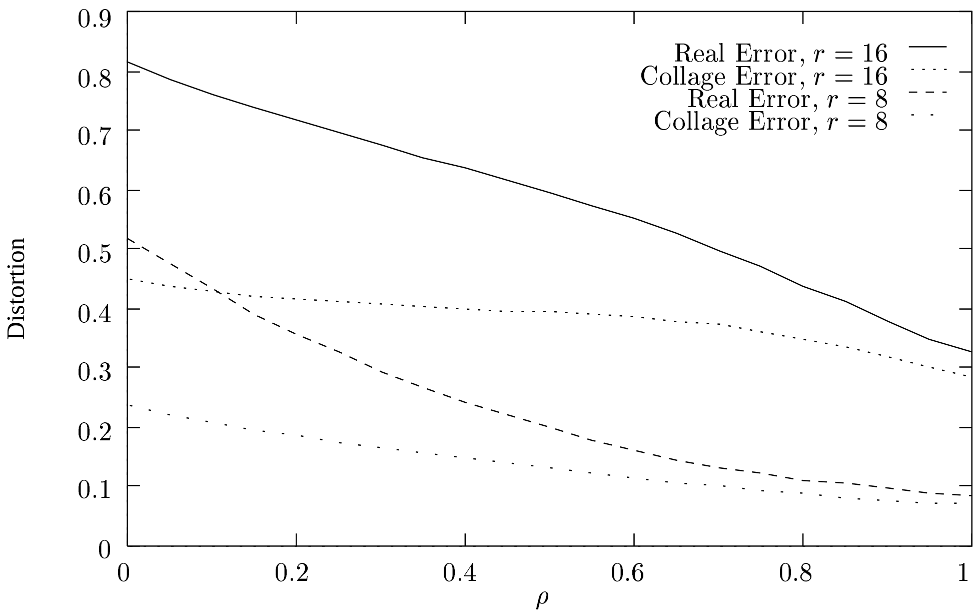

Figure 1: Real and Collage errors for $n = 4096$ and $s_{max} = 1.0$.

4 Contractivity

Since $s_{max} < 1$ is sufficient, but not necessary for contractivity [4], higher values are often employed. Figure 2 illustrates the decrease in collage errors for increasing s_{max}. It is clear that the dependence on s_{max} is considerably reduced for large ρ. However, if $s_{max} > 1.0$ convergence is not reliable for all ρ. As ρ is increased the maximum s_{max} resulting in reliable convergence is also increased (convergence is reliable for $\rho > 0.5$ when $s_{max} = 1.5$, and not at all when $s_{max} = 2.0$). The real error, in the region of convergence, also decreases with increasing s_{max}.

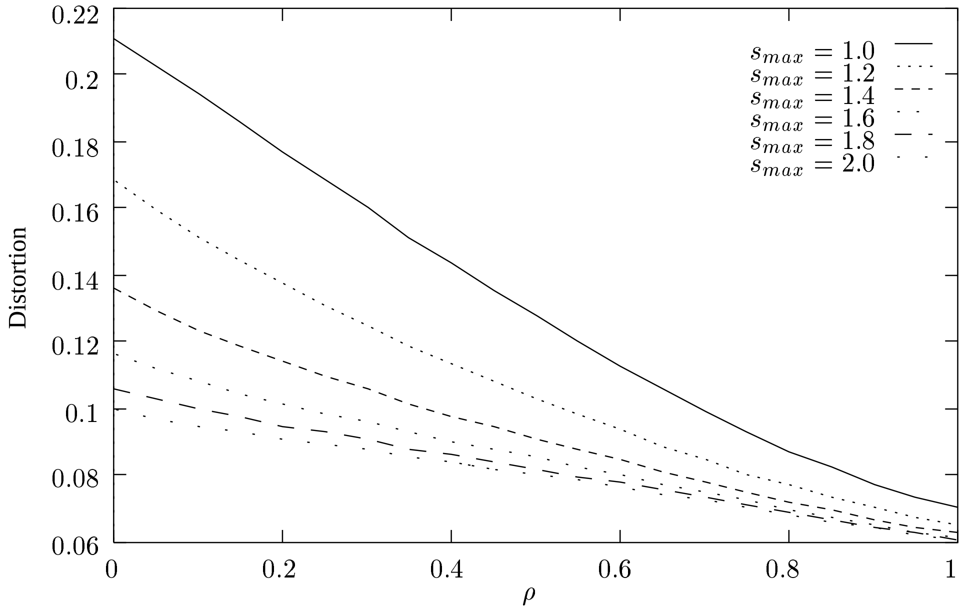

Figure 2: Collage errors for $n = 4096$ and $r = 8$.

5 Range Block Size

The increase in collage distortion with increasing range block size is clear from figure 3. Distortion decreases monotonically with increasing ρ for the smaller range blocks, while there is an intermediate maximum for the larger blocks (similar behaviour is observed for real errors). Similar curves are observed over a wide range of signal sizes. While the collage error is increased for smaller signals as a result of the reduction in domain pool size, and the positions of maxima vary, the dominant factor appears to be the correlation decay with distance in comparison with the range block size.

6 Distortion Rate Function

The distortion rate function for a first order Gauss-Markov model, where $R \geq \log_2(1 + \rho)$ (the small distortion region), is [3, App. D]

$$D(R) = (1 - \rho^2)2^{-2R}\sigma_x^2 = \sigma_z^2 2^{-2R}.$$

The lowest rate at which the small distortion requirement is satisfied for all ρ is $R = 1$, for which $D = 0.25$. On inspection of figure 3 it is clear that for block sizes larger than 12, the *collage* error with *unquantised* coefficients is considerably greater than the distortion limit, at this rate, for all ρ. However, since domain blocks are chosen from a pool of more than 2000 blocks, insufficient bits are available to code just the domain positions (neglecting the scaling

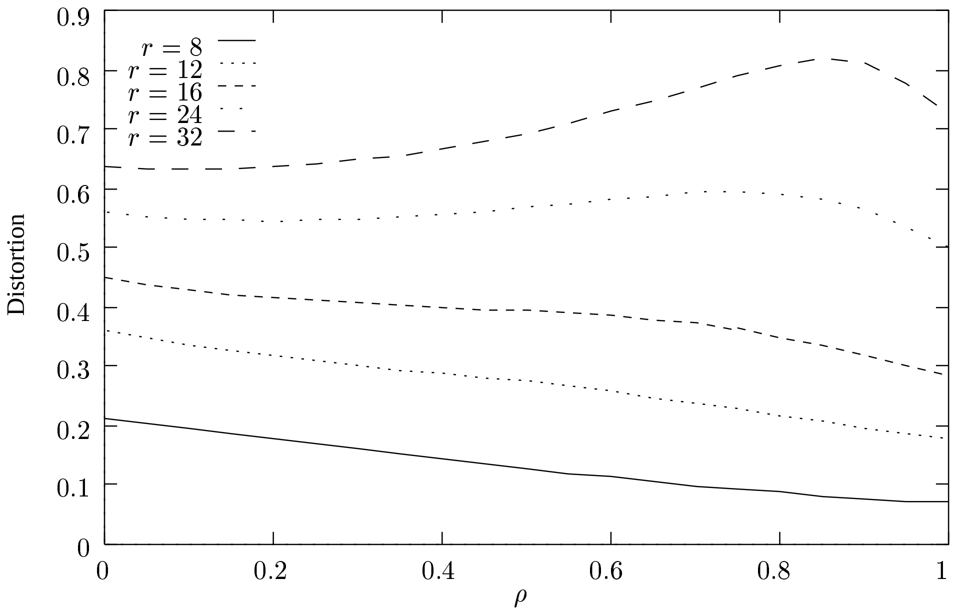

Figure 3: Collage errors for $n = 4096$ and $s_{max} = 1.0$.

and offset values) for range block sizes less than 12. Reducing the size of the domain pool by considering only neighbouring domain blocks for each range block reduces the number of bits required to specify domain position, but simultaneously increases the collage error beyond the distortion limit for unquantised scaling and offset coefficients.

Similar results were obtained for rates below 1 bit/sample. At higher rates the actual distortions obtained after quantising the fractal code coefficients were greater than optimum by more than a factor of 2 for the cases investigated. The best fractal coding performance in these experiments was obtained for $\rho \approx 0.8$, since the variance of the offset coefficient grows with increasing correlation, making it more difficult to code. In particular, performance was extremely poor (as a result of very high offset variance) for the $AR(1)$ model with $\rho = 1$, which generates stochastic fractal (Brownian motion) signals.

7 Conclusions

An evaluation of fractal coding performance such as that attempted here is hampered by the large number of free parameters such as block size and s_{max}. Nevertheless, our results indicate that fixed block size fractal coding of $AR(1)$ models is significantly suboptimal in a distortion-rate sense. While fractal coding has been compared with Vector Quantisation (VQ), there is an important distinction in that VQ may be adapted to any source statistics, whereas fractal coding represents an implicit source model (the assumption of "self-affinity"), and performs poorly when source statistics do not match those of this model.

References

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